VI. *Repulsion and Rotation produced by Alternating Electric Currents.*

By G. T. WALKER, *B.A., B.Sc., Fellow of Trinity College, Cambridge.*

Communicated by Professor J. J. THOMSON.

Received November 5,—Read December 10, 1891.

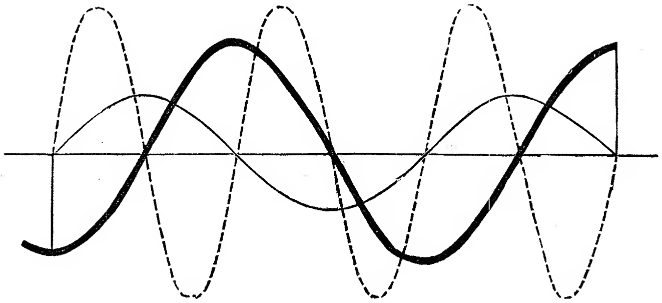
IN the 'Electrical World,' May, 1887, p. 258, or the 'Electrical Engineer' (New York), June, 1887, p. 211, "Novel Phenomena of Alternating Currents," may be seen an account of some experiments by Professor ELIHU THOMSON on the mechanical force between conductors in which alternating currents are circulating.

In the case of a ring of metal in the presence of an electromagnet, in the coils of which an alternating current is passing, a force of repulsion is experienced by the ring, and this may be accounted for in the words of Professor THOMSON as follows:—

"It may be stated as certainly true that were the induced currents in the closed conductor unaffected by any self-induction, the only phenomena exhibited would be alternate equal attractions and repulsions, because currents would be induced in opposite directions to that of the primary current when the latter current was changing from zero to maximum positive or negative current, so producing repulsion; and would be induced in the same direction when changing from maximum positive or negative to zero, so producing attractions."

This may be illustrated by fig. 1. Here the strong line represents the primary and the thin line the secondary, while of the dotted line any ordinate is the product of the ordinates of the lines representing the intensities of the currents and, hence, represents the mechanical force of attraction or repulsion.

Fig. 1.



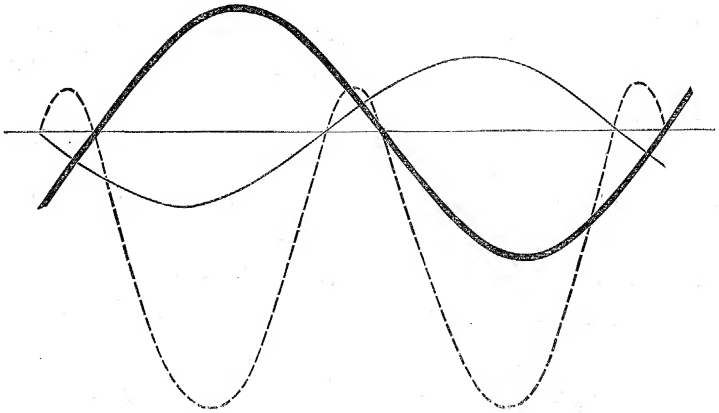
In the case of self-induction causing a lag, shift, or retardation of phase in the secondary current, the circumstances are described by Professor ELIHU THOMSON as follows :—

"It will be noticed that the period during which the currents are opposite, and during which repulsion can take place, is lengthened at the expense of the period during which the currents are in the right direction for attractive action.

"But far more important still in giving prominence to the repulsive effect than this difference of effective period, is the fact that, during the period of repulsion, both the induced and inducing currents have their greatest values, while, during the period of attraction, the currents are of small amounts comparatively. There is then a *repulsion due to the summative effects of strong opposite currents* for a *lengthened period* against an *attraction* due to the summative effects of *weak currents* of the *same direction* during a *shortened period*, the resultant effect being a greatly preponderating repulsion."

The diagram for this is given in fig. 2.

Fig. 2.



Professor THOMSON has proved experimentally that two circular coils, whose planes are perpendicular to the line joining their centres, repel one another when an alternating current traverses one of them.

If the coils consist of circular wires of radii A, a, and the planes be distant b, while the current traversing the primary is of strength I sin pt, then I have shown that the force of repulsion is

$$\mathrm{I}^2 \frac{2\pi^2 p^2 b \mathrm{N}}{\mathrm{S}^2 + p^2 \mathrm{N}^2} [2\mathrm{E} - (1 + \cos^2 \gamma)\, \mathrm{F}]\, [2\mathrm{F} - (1 + \sec^2 \gamma)\, \mathrm{E}],$$

where

$$\mathrm{S} = \text{resistance of secondary circuit},$$

$$\mathrm{N} = \text{its coefficient of self-induction},$$

$$\sin \gamma = \frac{2\sqrt{(\mathrm{A}a)}}{\sqrt{\{(\mathrm{A} + a)^2 + b^2\}}},$$

and F, E are complete elliptic integrals to modulus sin γ.

This repulsion may easily be taken advantage of by using it as the basis of a meter for alternating currents.

If the coils consist of two circles of radii a and c (the former the greater), with their centres coincident and planes making an angle θ, and we send a current $\mathrm{I} \sin pt$ through the larger, there will be a couple tending to increase the angle θ.

This has been proved experimentally by Professor THOMSON: its amount proves to be

$$\mathrm{I}^2 \frac{2\pi^4 p^2 \mathrm{N}}{\mathrm{S}^2 + p^2\mathrm{N}^2} \frac{c^4}{a^4}\left[\sin\theta\cos\theta + \tfrac{3}{8}\left(\frac{c}{a}\right)^2 \sin\theta\cos\theta\,(10\cos^2\theta - 3) + \ldots\right].$$

The positions $\theta = 0$ and $\theta = \pi/2$ are positions of equilibrium, the former being unstable and the latter stable.

By making the plane of the primary vertical and suspending the secondary inside so as to be capable of turning round a vertical axis by means of bifilar suspension with $\theta = 0$ as position of equilibrium, the deflection θ, when the alternating current is passing, will give the intensity of current.

We might also get the intensity by suspending the secondary by a single thread and observing the time of a small oscillation about $\theta = \pi/2$.

If the moment of inertia of the secondary about the vertical be mk^2 it is not difficult to show that the number of oscillations per second is

$$\mathrm{I}\,\frac{2\pi p c^2}{a\sqrt{\left\{\dfrac{2mk^2(\mathrm{S}^2 + \mathrm{N}^2p^3)}{\mathrm{N}}\right\}}}\left[1 - \tfrac{9}{8}\left(\frac{c}{a}\right)^2 \ldots\right].$$

Professor ELIHU THOMSON has devised other interesting experiments of which the following is an example :—

A sheet of copper is placed so as to half cover an alternating magnetic pole. Upon this, near the pole, is laid a hollow sphere of copper. The electromagnetic action produces a couple so powerful that the friction of rotation is overcome and the sphere is spun round.

The mathematical analysis for this case being complicated I have evaluated the couples called into action in various combinations of hollow spherical and cylindrical shells.

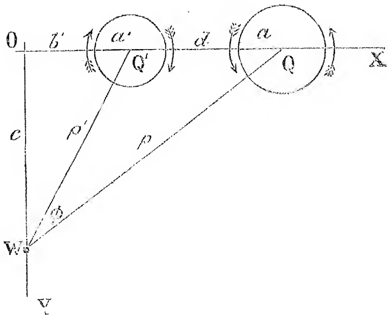
It is a known fact that in a spherical conductor no external field can give rise to induced currents that do not circulate in concentric spherical shells. After a preliminary theorem to the effect that there are no other families of surfaces which possess similar properties, the case has been considered of an infinitely long, thin, circular cylindrical shell in a field consisting of alternating currents parallel to its axis.

If the electrokinetic momentum of the primary field be expanded in harmonics over the cylinder, it turns out that if all the terms of each harmonic have the same phase, there will be no couple acting on the cylinder: as a particular case, if the external field have the same phase throughout there will be no couple.

The next case considered is that of two long cylindrical shells in the presence of an alternating current in a long parallel wire.

If the current in the wire be I cos pt, and σ, σ' be the resistances (across unit length of the surface) and a, a' the radii of the shells, while ρ, ρ' are their distances from the wire, d from one another, and c of the wire from the plane containing the two axes, then the couple on the a shell is in the direction represented in fig. 3, of amount

Fig. 3.



$$-\mathrm{I}^2\frac{8\pi^2p^2\sigma\sigma' a^3a'^3c}{\rho^2\rho'^2 d}\left[\frac{1}{\mathrm{D}_1\mathrm{D}'_1}+\frac{4a^2(\rho'^2-d^2)}{\mathrm{D}'_1\mathrm{D}_2\rho^2 d^2}+\ldots\right]$$

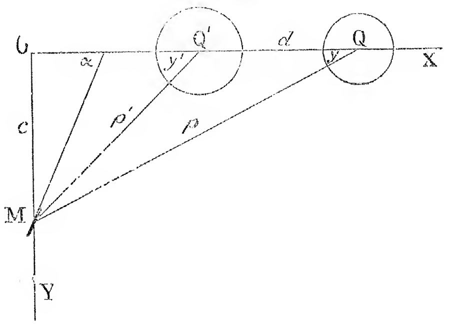
where

$$\mathrm{D}_n \equiv 4\pi^2a^2p^2+\sigma^2n^2,$$
$$\mathrm{D}'_n \equiv 4\pi^2a'^2p^2+\sigma'^2n^2.$$

In this case the couples on the two shells are, considering only the most important term, of equal magnitudes and of opposite signs.

If a second wire conveying a current of strength $-$ I cos pt be laid close to that already present, we obtain in effect a filament periodically magnetised in a direction

Fig. 4.



perpendicular to its length. Let this direction make α with the plane containing the axes. Then, if QQ′ (the line of length d) subtend ϕ at the magnet M, the couple on the a shell is

$$- A^2 \cdot \frac{8\pi^2 p^2 \sigma\sigma' a^3 a'^3}{d^2 D_1 D'_1 \rho^2 \rho'^2} \sin 2\phi,$$

where A cos pt is the intensity of the magnet and terms of the eighth degree in the radii are omitted.

Thus, to this order of approximation, we have the following results :—

(α.) The couples exerted are independent of α, *i.e.*, of the direction of the axis of the electromagnet.

(β.) The shells have equal couples in opposite directions, the parts of the shells directed towards one another being attracted towards the electromagnet if ϕ is less than a right angle, and driven away if ϕ exceeds a right angle.

Fig. 5.

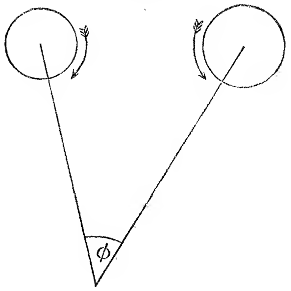
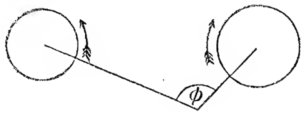


Fig. 6.



If ϕ be a right angle the couple is zero.

We next discuss the case of a thin spherical shell in any field and show that—

(α.) If the field be symmetrical round any diameter, there will be no couple about that diameter.

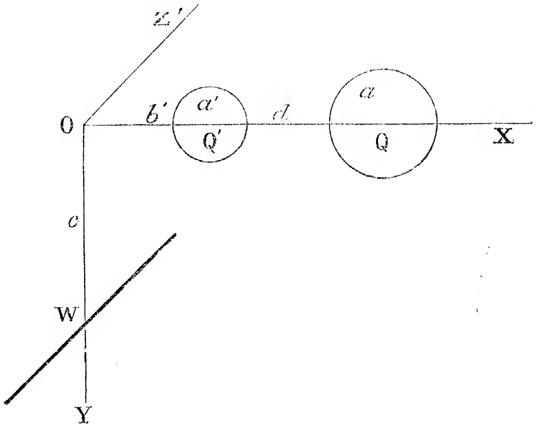
(β.) If the external field be completely in the same phase, or if, when the external magnetic potential is expanded in harmonics over the surface, the terms of the harmonic of any degree are in phases that are the same for the same degree, then the couple will vanish. From this it follows that if the primary field be in the same phase throughout, and any number of perfectly conducting bodies be introduced, their currents will be in the same phase as the primary field, and no couple will be produced.

It has been stated that in the case of Professor ELIHU THOMSON'S sphere, spinning on a sheet of copper, the effect was due to the sheet acting as a "shield" and producing an unsymmetrical field. That this explanation is not satisfactory will be evident on considering that the field is unsymmetrical before the sheet is interposed, and that the better the sheet conducts the better the shielding effect; so that if it be a perfect conductor a large couple would be expected whereas in reality there is none whatever.

The effect must, I think, be traced to the fact that the currents induced in the sheet are caused by self-induction to lag, so that the field in action on the sphere does not alternate in one phase.

The results obtained in the preceding analysis are applied to the case of two thin spherical shells in the presence of an alternating current in a straight infinite wire.

Fig. 7.



If the line joining the centres of the spheres be taken as OX, and the axis OY be the shortest distance between OX and the wire W (the last two lines being taken perpendicular to one another), then calling the distances of the centres of the spheres from the origin b, b', and their radii a, a' (as with the cylinders), we have as the first terms of the couples upon the a, b and the a', b' shells—

$$\frac{144\pi^2 p^2 \sigma\sigma' a^4 a'^4 c}{\rho^2\rho'^2\, d^3\, \Delta_1\, \Delta'_1}\,(2b + b')\,\mathrm{I}^2,$$

and

$$\frac{144\pi^2 p^2 \sigma\sigma' a^4 a'^4 c}{\rho^2\rho'^2\, d^3\, \Delta_1\, \Delta'_1}\,(b + 2b')\,\mathrm{I}^2,$$

where

$$\Delta_n \equiv 16\pi^2 a^2 p^2 + (2n + 1)^2\,\sigma^2,$$

and I cos pt, σ, σ', c, and d have the same meanings as before.

If we write h for $\frac{1}{2}(b + b')$, the couples are

$$\frac{72\pi^2 p^2 \sigma\sigma' a^4 a'^4 c}{\rho^2\rho'^2\, d^3 \Delta_1\, \Delta'_1}\,(6h \pm d)\,\mathrm{I}^2.$$

Hence

(α.) If σ, σ' or c vanish, the couples vanish, as might be expected.

(β.) The signs of the couples fall into three cases:—

(i.) When $6h$ is positive and greater than d, both couples are positive.

Fig. 8.

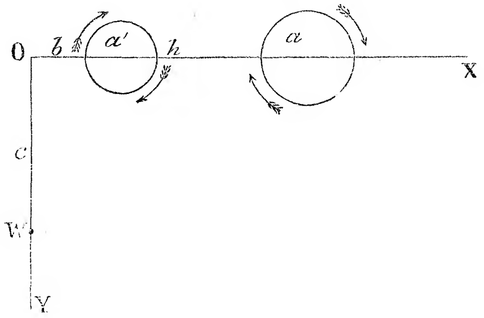
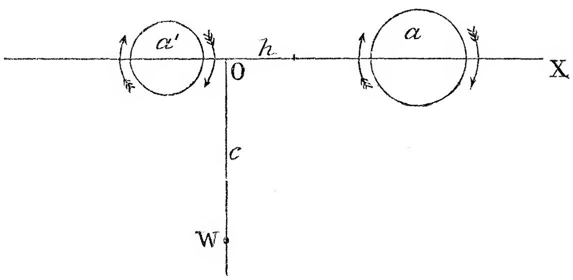
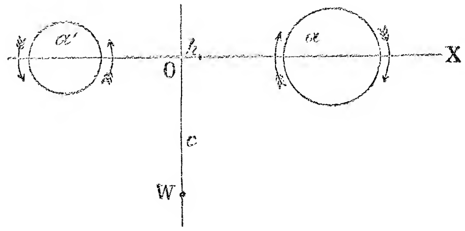


Fig. 9.



(ii.) When $6h$ is numerically less than d, the a, b couple is positive and the a', b' negative.

Fig. 10.



(iii.) When $6h$ is negative and numerically greater than d, both couples are negative.

Fig. 11.

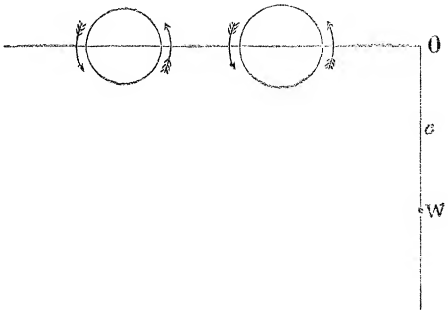
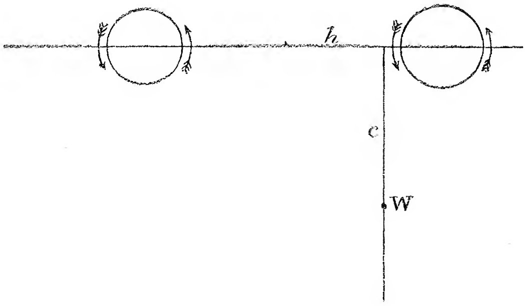


Fig. 12.



If we place another wire by the side of that already in position, and thus make a filament alternately magnetised in a direction perpendicular to its length, and making an angle α with OX (as with the cylinder), then the couple on the a, b sphere is

$$K^2 \frac{72\pi^2 p^2 a^4 a'^4 \sigma\sigma'}{\rho^2 \rho'^2 \Delta_1 \Delta'_1 d^3} [\sin 2(\gamma' - \gamma) + 3 \sin 2(\alpha - \gamma - \gamma')],$$

and on the a', b' sphere

$$K^2 \frac{72\pi^2 p^2 a^4 a'^4 \sigma\sigma'}{\rho^2 \rho'^2 \Delta_1 \Delta'_1 d^3} [- \sin 2(\gamma' - \gamma) + 3 \sin 2(\alpha - \gamma - \gamma')],$$

where K cos pt is the magnetic moment of unit length.

To this order then

(α.) If the couples on the shells be equal and opposite

$$\alpha - \gamma - \gamma' = 0, \text{ or } \tfrac{1}{2}\pi.$$

(β.) The couples will not vanish when $c = 0$ (and $\gamma = \gamma' = 0$) unless, in addition, $\alpha = 0$ or $\frac{1}{2}\pi$.

In this case considerations of symmetry would give right results.

(γ.) For an example, take

$$\gamma = 30^\circ, \qquad \gamma' = 60^\circ,$$

and the couples will be

$$K^2 \frac{72\pi^2 p^2 a^4 a'^4 \sigma\sigma'}{\rho^2 \rho'^2 \Delta_1 \Delta'_1 d^3} \left[\pm \frac{\sqrt{3}}{2} - 3 \sin 2\alpha \right],$$

the upper sign referring to the a, b shell.

The sign of the bracket will be + when $\alpha = -\frac{1}{4}\pi$.

" " " ± " $\alpha = 0$ or $\frac{1}{2}\pi$.

" " " − " $\alpha = +\frac{1}{4}\pi$.

Fig. 13.

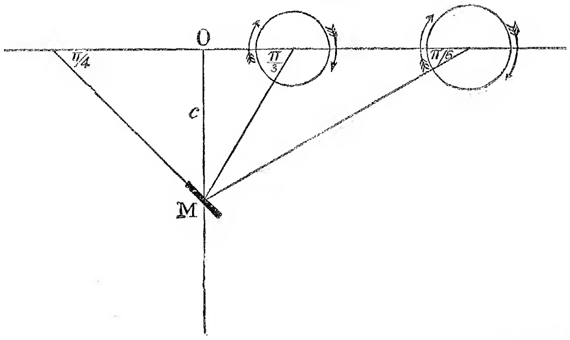


Fig. 14.

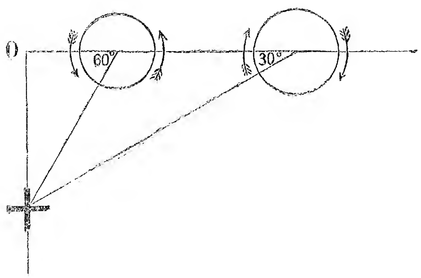
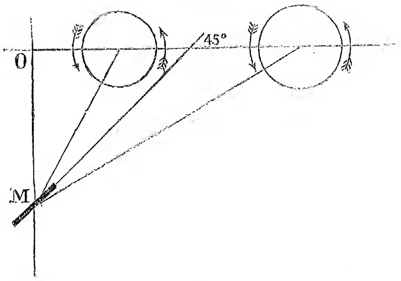


Fig. 15.



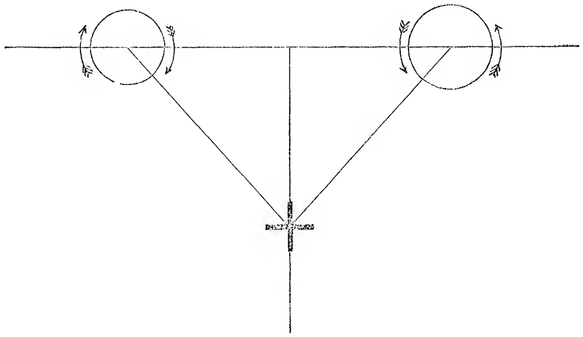
(δ.) In confirmation it may be noticed that when $\gamma = \pi - \gamma'$ (*i.e.*, when the spheres are equidistant from the origin), and α is zero or a right angle, the couples are equal and opposite.

Their magnitudes are

$$\mp \mathrm{K}^2 \frac{72\pi^2 p^2 a^4 a'^4 \sigma\sigma'}{\rho^2 \rho'^2 \Delta_1 \Delta'_1\, d^3} \sin 2\gamma,$$

and are negative and positive on the a, b and a', b' shells as long as the former is to the right of the latter.

Fig. 16.



We now consider the couples on two spherical shells in the presence of a magnetic pole of strength H cos pt.

We take the same axes as before with the magnetic pole at (o, c, o).

The couple on the a, b shell proves to be

$$- \mathrm{H}^2 \frac{36\pi^2 p^2 a^4 a'^4 c\sigma\sigma'}{\rho^3 \rho'^3 \Delta_1 \Delta'_1\, d^3} (b + 2b'),$$

and that on the a', b' shell

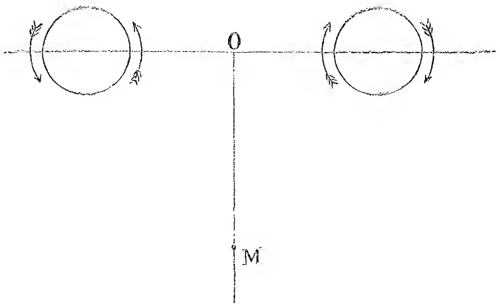
$$- \mathrm{H}^2 \frac{36\pi^2 p^2 a^4 a'^4 c\sigma\sigma'}{\rho^3 \rho'^3 \Delta_1 \Delta'_1\, d^3} (2b + b').$$

Hence

(α.) The couples vanish, as they should, when $c = 0$, or $\sigma = 0$, or $\sigma' = 0$.

(β.) When $b = - b'$ and $a = a'$, the figure is symmetrical to the plane YOZ, and the couples are equal and opposite: that on the a, b shell to the right is positive.

Fig. 17.



(γ.) For other cases the discussion is similar to that for the cylinders with parallel current I cos pt.

Thus (i.), when $h \{\equiv \frac{1}{2}(b+b')\}$ is positive, and greater than $d/6$ the couples are both negative.

Fig. 18.

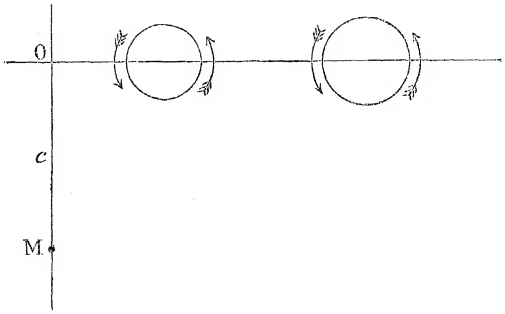
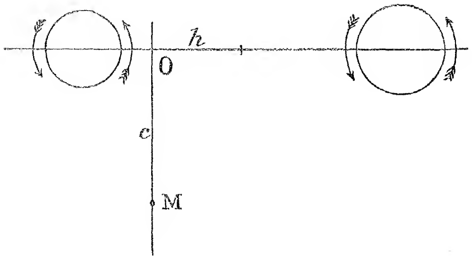
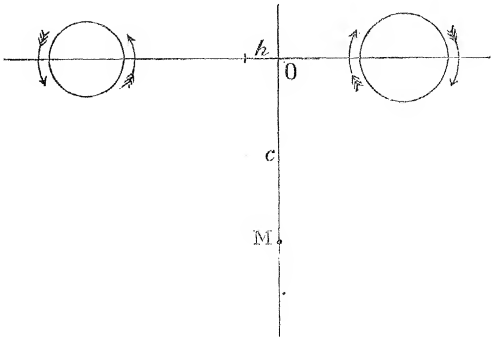


Fig. 19.



(ii.) When $6h$ is numerically less than d, the signs are $+$, $-$.

Fig. 20.



(iii.) When $6h$ is negative and numerically greater than d, the couples are both positive.

Fig. 21.

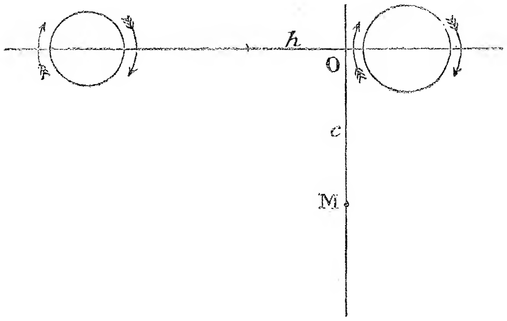
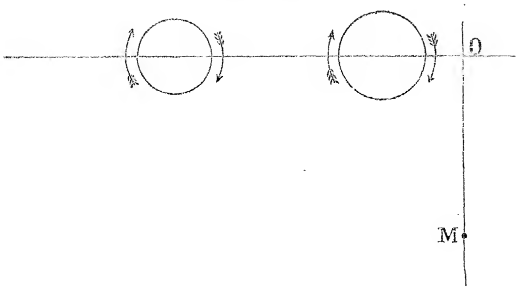


Fig. 22.



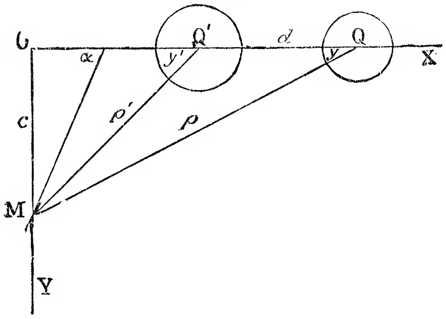
If we take two spherical shells in the presence of a small magnet of moment K cos pt, whose axis cuts the line joining the centres of the spheres at angle α, we find for the couple on the a, b shell

$$-\mathrm{K}^2 \frac{27\pi^2 p^2 a^4 a'^4 \sigma\sigma'}{2\Delta_1 \Delta'_1 d^3 \rho^3 \rho'^3} \phi(\alpha, \gamma, \gamma'),$$

where $\phi(\alpha, \gamma, \gamma')$ is written for

$$\sin 2\alpha + 3\,(\sin 2\gamma + \sin 2\gamma') + \sin \overline{2\alpha - 2\gamma'} - \sin \overline{2\alpha - 2\gamma}$$
$$- 3 \sin \overline{2\gamma' - 2\gamma} - 9 \sin \overline{2\alpha - 2\gamma - 2\gamma'}.$$

Fig. 23.



For the other shell, we interchange γ and γ'.

For the couples to be equal and opposite when $\gamma' = \pi - \gamma$, we must have $\alpha = 0$ or $\frac{1}{2}\pi$. When $\gamma = \frac{1}{3}\pi$, the a, b couple is positive for both values of α, and when $\gamma = \frac{1}{6}\pi$, it is negative.

Fig. 24.

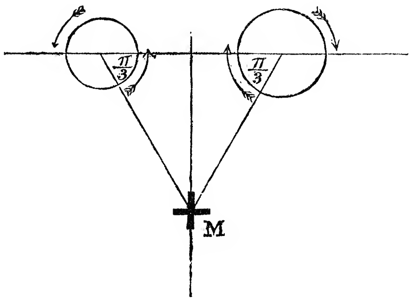
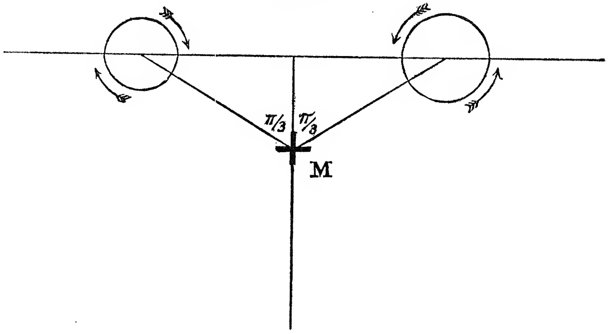


Fig. 25.



We now take as an example

$$\gamma = \pi/6, \qquad \gamma' = \pi/3,$$

and the a, b couple is negative from 0° to about 98° 23′, positive thence to 171° 37′, and negative to 180°. The a', b' couple is negative from 0° to 112° 34′, positive thence to 157° 26′ and negative to 180°.

In fig. 26, $\alpha = 45°$; in fig. 27, $\alpha = 105°$.

Fig. 26.

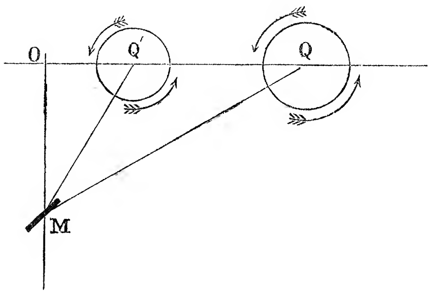
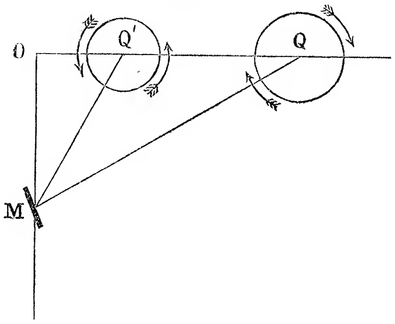


Fig. 27.



In fig. 28, $\alpha = 135°$; in fig. 29, $\alpha = 165°$.

Fig. 28.

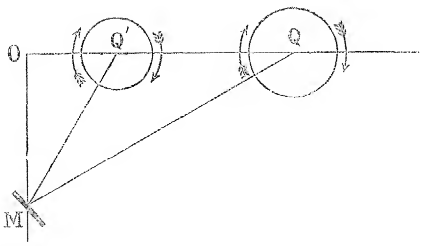
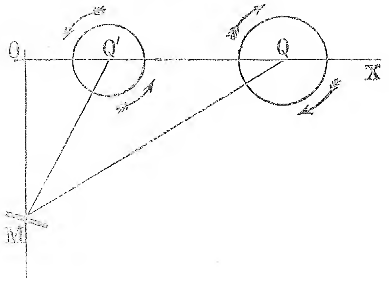


Fig. 29.



For general principles I have derived assistance from the paper by Professor C. Niven, in the 'Philosophical Transactions' of 1881, "On the Induction of Electric Currents in Infinite Plates and Spherical Shells."

The Mechanical Effects of the Currents Induced in One Coil by those in Another.

1. Adopting Maxwell's notation and taking $\dot{y}_1$, $\dot{y}_2$, as the currents in the two circuits, we have

$$2T = L\dot{y}_1^2 + 2M\dot{y}_1\dot{y}_2 + N\dot{y}_2^2.$$

If R and S be the resistance of the primary and secondary circuits, and the current in the former be I sin pt, then for the second circuit

$$\frac{d}{dt}(M\dot{y}_1 + N\dot{y}_2) + S\dot{y}_2 = 0,$$

whence

$$\dot{y}_2 = A \cos pt + B \sin pt,$$

where

$$\frac{A}{SMp} = \frac{B}{MNp^2} = \frac{-I}{p^2N^2 + S^2}.$$

Thus the "lag" due to self-induction in the secondary coil is $\tan^{-1} Np/S$, and the electromagnetic force tending to increase a coordinate x is

$$\dot{y}_1\dot{y}_2\frac{dM}{dx},$$

the mean value of which is

$$-\frac{1}{2}\frac{I^2MNp^2}{S^2 + N^2p^2}\frac{dM}{dx}.$$

If the coils consist of circular wires of radii A and a, and their planes be perpendicular to the line of length b that joins their centres, then Maxwell shows (§ 701) that

$$\mathrm{M} = \frac{4\pi\sqrt{(\mathrm{A}a)}}{\sin\gamma}\{-(1+\cos^2\gamma)\,\mathrm{F} + 2\mathrm{E}\},$$

where

$$\sin\gamma = \frac{2\sqrt{(\mathrm{A}a)}}{\sqrt{\{(\mathrm{A}+a)^2 + b^2\}}},$$

and F and E are complete elliptic integrals to modulus $\sin\gamma$.

Also

$$\frac{d\mathrm{M}}{db} = -\frac{\pi}{\sqrt{(\mathrm{A}a)}}\, b\sin\gamma\,\{2\mathrm{F} - (1+\sec^2\gamma)\,\mathrm{E}\},$$

so that the repulsion is

$$\mathrm{I}^2\,\frac{2\pi^2 p^2 b\mathrm{N}}{\mathrm{S}^2 + p^2\mathrm{N}^2}\,[2\mathrm{E} - (1+\cos^2\gamma)\,\mathrm{F}]\,[2\mathrm{F} - (1+\sec^2\gamma)\,\mathrm{E}].$$

If the coils consist of two circles of radii a and c (the former the greater) with their centres coincident and planes inclined θ, we have from Maxwell (§ 697),

$$\begin{aligned}\mathrm{M} &= 4\pi^2 c\left\{\frac{1}{1.2}\,\frac{c}{a}\,[\mathrm{P}'_1(0)]^2\,\mathrm{P}_1(\theta) + \ldots + \frac{1}{r(r+1)}\left(\frac{c}{a}\right)^r [\mathrm{P}'_r(0)]^2\,\mathrm{P}_r(\theta) + \ldots\right\}\\ &= 4\pi^2 c\left\{\frac{1}{1.2}\,\frac{c}{a}\,\mathrm{P}_1(\theta) + \frac{1}{3.4}\left(\frac{c}{a}\right)^3\left(\frac{3}{2}\right)^2\mathrm{P}_3(\theta) + \ldots\right.\\ &\qquad\left. + \frac{1}{r.r+1}\left(\frac{c}{a}\right)^r\left[\frac{3.5\ldots r}{2.4\ldots(r+1)}\right]^2\mathrm{P}_r(\theta)\ldots\right\}\end{aligned}$$

where r must now be odd.

The couple tending to increase θ is

$$-\tfrac{1}{2}\mathrm{I}^2\,\frac{\mathrm{MN}p^2}{\mathrm{S}^2 + p^2\mathrm{N}^2}\,\frac{d\mathrm{M}}{d\theta},$$

or

$$\begin{aligned}&\tfrac{1}{2}\mathrm{I}^2\,\frac{\mathrm{N}p^2}{\mathrm{S}^2+p^2\mathrm{N}^2}\,16\pi^4 c^2\left[\frac{1}{1.2}\,\frac{c}{a}\cos\theta + \frac{1}{3.4}\left(\frac{c}{a}\right)^3\left(\frac{3}{2}\right)^2\frac{5\cos^3\theta - 3\cos\theta}{2} + \ldots\right]\\ &\qquad\times\left[\frac{1}{1.2}\,\frac{c}{a} + \frac{1}{3.4}\left(\frac{c}{a}\right)^3\left(\frac{3}{2}\right)^2\frac{15\cos^2\theta - 3}{2} + \ldots\right]\sin\theta,\end{aligned}$$

or

$$\mathrm{I}^2\,\frac{2\pi^4 p^2\mathrm{N}c^2}{\mathrm{S}^2+p^2\mathrm{N}^2}\,\frac{c^2}{a^2}\,\sin\theta\cos\theta\left[1 + \tfrac{3}{8}\left(\frac{c}{a}\right)^2(10\cos^2\theta - 3) + \ldots\right].$$

2. In the course of the following work it will often be necessary to know what kind of distribution of electric currents is likely to be set up in conductors of various shapes on the introduction of external fields: it is known that in a sphere no external field can give rise to currents that do not circulate in concentric spherical surfaces, and it

might be thought that for some other surface an analogous property held, as that in an anchor ring the induced currents always lay in toroidal surfaces.

The question may be stated as follows :—

Given a system of orthogonal surfaces,

$$\begin{aligned} a &= \text{constant}, \\ b &= \text{constant}, \\ c &= \text{constant}, \end{aligned}$$

and a uniform conductor whose bounding surface has a constant, what is the condition that, whatever be the nature of the external field, the induced currents may lie in the a surfaces.

If the length δs of the line joining the consecutive points a, b, c, and $a + \delta a$, $b + \delta b$, $c + \delta c$ be given by

$$\delta s^2 = \mathrm{A}^2\,\delta a^2 + \mathrm{B}^2\,\delta b^2 + \mathrm{C}^2\,\delta c^2,$$

and if

$$\begin{array}{ccc} u, & v, & w, \\ \alpha, & \beta, & \gamma, \\ \mathrm{F}, & \mathrm{G}, & \mathrm{H}, \end{array}$$

denote the components of electric current, of magnetic force, and of electromagnetic momentum along the normals to the three orthogonal surfaces through any point, then MAXWELL's equations of electric currents become

$$\begin{aligned} 4\pi\mathrm{BC}\,.\,u &= \frac{\partial}{\partial b}(\mathrm{C}\gamma) - \frac{\partial}{\partial c}(\mathrm{B}\beta) \\ 4\pi\mathrm{CA}\,.\,v &= \frac{\partial}{\partial c}(\mathrm{A}\alpha) - \frac{\partial}{\partial a}(\mathrm{C}\gamma) \\ 4\pi\mathrm{AB}\,.\,w &= \frac{\partial}{\partial a}(\mathrm{B}\beta) - \frac{\partial}{\partial b}(\mathrm{A}\alpha), \end{aligned}$$

and if there be no magnetisable matter in the conductor, α, β, γ are components of magnetic induction, and are given by

$$\mathrm{BC}\alpha = \frac{\partial}{\partial b}(\mathrm{CH}) - \frac{\partial}{\partial c}(\mathrm{BG}),$$

with two similar equations.

If σ denote the specific resistance of the conductor, and ψ the electrostatic potential, then within the conductor we have (it being at rest)

$$\sigma u = -\frac{\partial \mathrm{F}}{\partial t} - \frac{\partial \psi}{\mathrm{A}\,\partial a},$$

with two similar equations.

Remembering that the surface has α constant over it, we have as the surface condition (Professor C. NIVEN, 'Phil. Trans.,' 1881, p. 313)

$$\frac{\partial F}{\partial t} + \frac{\partial \psi}{A\,\partial \alpha} = 0,$$

the normal outwards being in the direction of α increasing: and (from the same source) within the conductor

$$\nabla^2 \psi = 0.$$

Hence $\partial\psi/\partial n$ being known, ψ is uniquely determined (MAXWELL, vol. 5, § 100, *e*).

Let us write $\psi \equiv \partial\chi/\partial t$, and

$$F' \equiv F + \frac{\partial \chi}{A\,\partial \alpha}$$

$$G' \equiv G + \frac{\partial \chi}{B\,\partial b}$$

$$H' \equiv H + \frac{\partial \chi}{C\,\partial c},$$

then our equations are typified by

$$\left.\begin{aligned} \sigma u &= -\frac{\partial F'}{\partial t} \\ BC\alpha &= \frac{\partial}{\partial b}(CH') - \frac{\partial}{\partial c}(BG') \\ 4\pi u BC &= \frac{\partial}{\partial b}(C\gamma) - \frac{\partial}{\partial c}(B\beta) \end{aligned}\right\}$$

If now the conductor be such that no external field can give rise to currents normal to α surfaces, then for all values of F′, G′, H′ consistent with

$$\frac{\partial}{\partial t} F' = 0$$

at the surface, and

$$\frac{\partial}{\partial \alpha}(BCF') + \frac{\partial}{\partial b}(CAG') + \frac{\partial}{\partial c}(ABH') = 0$$

within the conductor, u must everywhere vanish: that is, inside,

$$\left.\begin{aligned} 0 &= \frac{\partial F'}{\partial t} \\ 0 &= \frac{\partial}{\partial b}(C\gamma) - {}^{\partial}\ (B\beta) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (i.).$$

Now F′ is everywhere zero before the external field is brought into existence, hence by the former equation (1) it is everywhere zero permanently.

On substituting in the latter equation from the equations connecting β, γ with $\mathrm{F}', \mathrm{G}', \mathrm{H}'$, and remembering that $\mathrm{F}' = 0$, we get

$$\frac{\partial}{\partial b}\left\{\frac{\mathrm{C}}{\mathrm{AB}}\frac{\partial}{\partial a}(\mathrm{BG}')\right\} + \frac{\partial}{\partial c}\left\{\frac{\mathrm{B}}{\mathrm{AC}}\frac{\partial}{\partial a}(\mathrm{CH}')\right\} = 0,$$

and this is to hold for all values of G', H' consistent with

$$\frac{\partial}{\partial b}(\mathrm{CAG}') + \frac{\partial}{\partial c}(\mathrm{ABH}') = 0.$$

This condition is replaced by taking

$$\mathrm{CAG}' \equiv \frac{\partial f}{\partial c}$$

$$\mathrm{ABH}' \equiv -\frac{\partial f}{\partial b},$$

and the former relation becomes

$$\frac{\partial}{\partial b}\left[\frac{\mathrm{C}}{\mathrm{AB}}\frac{\partial}{\partial a}\left(\frac{\mathrm{B}}{\mathrm{AC}}\frac{\partial f}{\partial c}\right)\right] - \frac{\partial}{\partial c}\left[\frac{\mathrm{B}}{\mathrm{AC}}\frac{\partial}{\partial a}\left(\frac{\mathrm{C}}{\mathrm{AB}}\frac{\partial f}{\partial b}\right)\right] \equiv 0,$$

or

$$\frac{\partial}{\partial b}\left[\frac{\mathrm{C}}{\mathrm{AB}}f_c\frac{\partial}{\partial a}\left(\frac{\mathrm{B}}{\mathrm{AC}}\right) + \frac{1}{\mathrm{A}^2}f_{ac}\right] - \frac{\partial}{\partial c}\left[\frac{\mathrm{B}}{\mathrm{AC}}f_b\frac{\partial}{\partial a}\left(\frac{\mathrm{C}}{\mathrm{AB}}\right) + \frac{1}{\mathrm{A}^2}f_{ab}\right] \equiv 0.$$

therefore

$$f_{bc}\left[\frac{\mathrm{C}}{\mathrm{AB}}\frac{\partial}{\partial a}\left(\frac{\mathrm{B}}{\mathrm{AC}}\right) - \frac{\mathrm{B}}{\mathrm{AC}}\frac{\partial}{\partial b}\left(\frac{\mathrm{C}}{\mathrm{AB}}\right)\right] + f_c\frac{\partial}{\partial b}\left[\frac{\mathrm{C}}{\mathrm{AB}}\frac{\partial}{\partial a}\left(\frac{\mathrm{B}}{\mathrm{AC}}\right)\right] - f_b\frac{\partial}{\partial c}\left[\frac{\mathrm{B}}{\mathrm{AC}}\frac{\partial}{\partial a}\left(\frac{\mathrm{C}}{\mathrm{AB}}\right)\right]$$
$$+ f_{ac}\frac{\partial}{\partial b}\left(\frac{1}{\mathrm{A}^2}\right) - f_{ab}\frac{\partial}{\partial c}\left(\frac{1}{\mathrm{A}^2}\right) \equiv 0.$$

Now f being arbitrary the coefficients of f_{bc}, &c. must vanish, therefore

$$\frac{\mathrm{C}}{\mathrm{AB}}\frac{\partial}{\partial a}\left(\frac{\mathrm{B}}{\mathrm{AC}}\right) = \frac{\mathrm{B}}{\mathrm{AC}}\frac{\partial}{\partial a}\left(\frac{\mathrm{C}}{\mathrm{AB}}\right) = \text{function of } a \text{ (or constant)} \quad . \quad . \quad . \quad . \quad \text{(ii.)},$$

and

$$\frac{1}{\mathrm{A}^2} = \text{function of } a \text{ (or constant)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(iii.)}.$$

On differentiation, the relation

$$\frac{\mathrm{C}}{\mathrm{AB}}\frac{\partial}{\partial a}\left(\frac{\mathrm{B}}{\mathrm{AC}}\right) = \frac{\mathrm{B}}{\mathrm{AC}}\frac{\partial}{\partial a}\left(\frac{\mathrm{C}}{\mathrm{AB}}\right)$$

becomes

$$\frac{\mathrm{C}}{\mathrm{A}^2\mathrm{B}}\frac{\partial}{\partial a}\left(\frac{\mathrm{B}}{\mathrm{C}}\right) + \frac{1}{\mathrm{A}}\frac{\partial}{\partial a}\left(\frac{1}{\mathrm{A}}\right) = \frac{\mathrm{B}}{\mathrm{A}^2\mathrm{C}}\frac{\partial}{\partial a}\left(\frac{\mathrm{C}}{\mathrm{B}}\right) + \frac{1}{\mathrm{A}}\frac{\partial}{\partial a}\left(\frac{1}{\mathrm{A}}\right).$$

Therefore

$$\frac{\mathrm{C}}{\mathrm{B}}\frac{\partial}{\partial a}\left(\frac{\mathrm{B}}{\mathrm{C}}\right) - \frac{\mathrm{B}}{\mathrm{C}}\frac{\partial}{\partial a}\left(\frac{\mathrm{C}}{\mathrm{B}}\right) = 0.$$

Therefore

$$\frac{\partial}{\partial a}\left(\frac{\mathrm{B}}{\mathrm{C}}\right) = 0.$$

Hence B/C is a function of b, c only.

Also it was proved that

$$\frac{\mathrm{C}}{\mathrm{AB}}\frac{\partial}{\partial a}\left(\frac{\mathrm{B}}{\mathrm{AC}}\right) = \text{function of } a \text{ (or constant)}.$$

Therefore

$$\frac{1}{\mathrm{A}}\frac{\partial}{\partial a}\left(\frac{1}{\mathrm{A}}\right) = \text{function of } a \text{ (or constant)}.$$

This, as well as the last condition (iii.), is satisfied by A = function of a. Geometrically this means that the normal distance between a and $a + \delta a$ is everywhere the same, *i.e.*, that the a surfaces are parallel.

Now the radius of curvature ρ_{ab} of the normal section of a constant along the normal to b is given by

$$\frac{1}{\rho_{ab}} = \frac{1}{\mathrm{AB}}\frac{\partial \mathrm{B}}{\partial a},$$

and so too

$$\frac{1}{\rho_{ac}} = \frac{1}{\mathrm{AC}}\frac{\partial \mathrm{C}}{\partial a}.$$

Hence, if B/C is independent of a,

$$\rho_{ab} = \rho_{ac},$$

i.e., the two curvatures at each point of an a surface are equal.

Thus, the property in question is satisfied by a sphere, and only by a sphere or spherical shell, including, as a special case, a plane slab infinite in both directions.

The Mechanical Effect of Currents set up in a Thin Circular Cylindric Shell of Infinite Length.

3. We consider the case of a periodic field consisting of currents parallel to the axis of the shell.

In this case the surface condition informs us that since there is no normal component of electromagnetic momentum, there will be no electrostatic potential; also, if the axis of the cylinder be taken as the axis of cylindric coordinates, we have

$$\sigma w = -\frac{d}{dt}(\mathrm{H} + \mathrm{H}_0) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(iv.)},$$

where H_0 is the momentum due to the external field and H that due to the induced currents, while σ and w represent the resistance and current across unit length of the surface.

Consider

$$\mathrm{H}_0 = e^{in\theta} \sin pt$$

at the surface of the shell, and let the induced currents give rise to momentum at the surface of

$$\mathrm{H} = e^{in\theta} [\mathrm{B} \sin pt + \mathrm{C} \cos pt].$$

Since H is the potential of a distribution on the cylinder of imaginary matter of surface density w, we have

$$w = \frac{n}{2\pi a} e^{in\theta} [\mathrm{B} \sin pt + \mathrm{C} \cos pt].$$

On substituting in (iv.) we see that

$$\frac{\sigma n}{2\pi a} [\mathrm{B} \sin pt + \mathrm{C} \cos pt] \equiv - p [\mathrm{B} \cos pt - \mathrm{C} \sin pt] - p\mathrm{A} \cos pt.$$

Therefore

$$\left.\begin{aligned} \frac{\sigma n}{2\pi a} \mathrm{B} - p\mathrm{C} &= 0 \\ p\mathrm{A} + p\mathrm{B} + \frac{\sigma n}{2pa} \mathrm{C} &= 0. \end{aligned}\right\}$$

Therefore

$$\frac{-2\pi ap\mathrm{A}}{4\pi^2a^2p^2 + \sigma^2n^2} = \frac{\mathrm{B}}{2\pi ap} = \frac{\mathrm{C}}{\sigma n}.$$

Hence

$$w = - \mathrm{A} \frac{np\,(2\pi ap \sin pt + \sigma n \cos pt)}{4\pi^2a^2p^2 + \sigma^2n^2}\, e^{in\theta}$$

corresponding to a term

$$\mathrm{H}_0 = \mathrm{A}e^{in\theta} \sin pt.$$

On differentiating with respect to t, we see that if

$$\mathrm{H}_0 = \mathrm{A}'e^{in\theta} \cos pt,$$

then

$$w = - \mathrm{A}' \frac{np\,(2\pi ap \cos pt - \sigma n \sin pt)}{4\pi^2a^2p^2 + \sigma^2n^2}\, e^{in\theta}.$$

On separating real and imaginary parts, it is clear that when

$$\begin{aligned} \mathrm{H}_0 = {} & [\mathrm{M} \cos n\theta + \mathrm{N} \sin n\theta] \sin pt \\ & + [\mathrm{Q} \cos n\theta + \mathrm{R} \sin n\theta] \cos pt, \end{aligned}$$

then at the surface

$$H = -\frac{2\pi ap}{4\pi^2a^2p^2+\sigma^2n^2}[2\pi ap\sin pt+\sigma n\cos pt][M\cos n\theta+N\sin n\theta]$$

$$-\frac{2\pi ap}{4\pi^2a^2p^2+\sigma^2n^2}[2\pi ap\cos pt-\sigma n\sin pt][Q\cos n\theta+R\sin n\theta]. \quad . \quad . \quad (\text{v}),$$

and

$$w = \frac{n}{2\pi a}H.$$

4. We now proceed to find the components of magnetic force. When there is no magnetic matter (as we assume to be the case) the equations in cylindricals connecting α, β, γ with F, G, H, are

$$\varpi\alpha = \frac{\partial H}{\partial\theta} - \varpi\frac{\partial G}{\partial z},$$

$$\beta = \frac{\partial F}{\partial z} - \frac{\partial H}{\partial\varpi},$$

$$\varpi\gamma = \frac{\partial}{\partial\varpi}(\varpi G) - \frac{\partial F}{\partial\theta}.$$

With the distribution that we have chosen, $F = 0$, $G = 0$, and hence

$$\left.\begin{aligned}\alpha &= \frac{1}{\varpi}\frac{\partial H}{\partial\theta}\\ \beta &= -\frac{\partial H}{\partial\varpi}\\ \gamma &= 0\end{aligned}\right\}.$$

When the value of H_0 at the surface is

$$H_0 = Ae^{in\theta}\sin pt,$$

then just outside,

$$H_0 = A\left(\frac{\varpi}{a}\right)^n e^{in\theta}\sin pt,$$

so that just outside,

$$\left.\begin{aligned}\alpha_0 &= inA\frac{\varpi^{n-1}}{a^n}e^{in\theta}\sin pt\\ \beta_0 &= -nA\frac{\varpi^{n-1}}{a^n}e^{in\theta}\sin pt\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{vi.}).$$

Now when we consider the mechanical effects, since, on AMPÈRE's theory the mechanical effects of two elements of currents upon each other are equal and oppositely directed in the straight line joining them, the field, due to the induced currents, will have no direct resultant mechanical effect on the cylinder, and it will suffice to take the components of induction due to the external system.

The components of electromagnetic force along $\delta\varpi$, $\varpi\,\delta\theta$, δz are

$$-\beta w, \quad \alpha w, \quad 0,$$

and hence the mechanical force on the shell, parallel to the initial line from which θ is measured, is (per unit length)

$$a\int_0^{2\pi}\delta\theta\,.\,w\,(-\beta\cos\theta-\alpha\sin\theta).$$

The force in the perpendicular direction (perpendicular to the axis of the shell) is

$$a\int_0^{2\pi}\delta\theta\,.\,w\,(\alpha\cos\theta-\beta\sin\theta).$$

The couple about the axis is

$$a^2\int_0^{2\pi}\delta\theta\,.\,\alpha w$$

tending to increase θ.

Now, with the kind of field that we have taken,

$$\mathrm{F}_0=0,$$
$$\mathrm{G}_0=0,$$

and the period being $2\pi/p$, the value of H_0 over the surface may be expanded in the series,

$$\mathrm{H}_0=\sum_{n=0}^{n=\infty}\{[\mathrm{M}_n\cos n\theta+\mathrm{N}_n\sin n\theta]\sin pt+[\mathrm{Q}_n\cos n\theta+\mathrm{R}_n\sin n\theta]\cos pt\}.$$

We have already shown (vi.) that when

$$\mathrm{H}_0=\mathrm{A}e^{in\theta}\sin pt$$

at the surface, then at the surface

$$\alpha_0=\frac{in}{a}\mathrm{A}e^{in\theta}\sin pt.$$

Hence, with the expansion of H_0, we get

$$\alpha_0=\sum_{\substack{n=0\\ \text{or } 1}}^{n=\infty}\frac{n}{a}\{[-\mathrm{M}_n\sin n\theta+\mathrm{N}_n\cos n\theta]\sin pt+[-\mathrm{Q}_n\sin n\theta+\mathrm{R}_n\cos n\theta]\cos pt\}.$$

The corresponding value of w has been found to be (v.)

$$\sum_{n=1}^{n=\infty}\frac{-np}{4\pi^2a^2p^2+\sigma^2n^2}\{[2\pi ap\sin pt+\sigma n\cos pt][\mathrm{M}_n\cos n\theta+\mathrm{N}_n\sin n\theta]$$
$$+[2\pi ap\cos pt-\sigma n\sin pt][\mathrm{Q}_n\cos n\theta+\mathrm{R}_n\sin n\theta]\}$$

In finding the mean value of the couple, $a^2\int_0^{2\pi} \alpha_0 w\,\delta\theta$, we bear in mind that the mean values of $\sin^2 pt$ and of $\cos^2 pt$ are $\frac{1}{2}$, and of $\sin pt \,.\, \cos pt$, zero. On picking out the coefficients of $\sin^2 pt$ and $\cos^2 pt$ in $\alpha_0 w$, we find for the mean value of the couple

$$\int_0^{2\pi}\delta\theta \sum_{n=1}^{n=\infty} \frac{-an^2p}{2(4\pi^2a^2p^2+\sigma^2n^2)} \left[\begin{array}{l}(-\mathrm{M}_n\sin n\theta+\mathrm{N}_n\cos n\theta)(2\pi ap\{\mathrm{M}_n\cos n\theta+\mathrm{N}_n\sin n\theta\} \\ \qquad\qquad -\sigma n\{\mathrm{Q}_n\cos n\theta+\mathrm{R}_n\sin n\theta\}) \\ +(-\mathrm{Q}_n\sin n\theta+\mathrm{R}_n\cos n\theta)(\sigma n\{\mathrm{M}_n\cos n\theta+\mathrm{N}_n\sin n\theta\} \\ \qquad\qquad +2\pi ap\{\mathrm{Q}_n\cos n\theta+\mathrm{R}_n\sin n\theta\}).\end{array}\right]$$

Also

$$\int_0^{2\pi}\sin^2 \mathrm{n}\theta\,\delta\theta = \int_0^{2\pi}\cos^2 n\theta\,\delta\theta = \pi,$$

and

$$\int_0^{2\pi}\sin n\theta\cos n\theta\,\delta\theta = 0.$$

Thus the mean value of the couple becomes

$$\sum_1^{\infty}\frac{\pi an^2p}{2(4\pi^2a^2p^2+\sigma^2n^2)}\left[\begin{array}{l}2\pi ap\{-\mathrm{M}_n\mathrm{N}_n+\mathrm{M}_n\mathrm{N}_n-\mathrm{Q}_n\mathrm{R}_n+\mathrm{Q}_n\mathrm{R}_n\} \\ \qquad +\sigma n\{\mathrm{Q}_n\mathrm{N}_n-\mathrm{M}_n\mathrm{R}_n-\mathrm{M}_n\mathrm{R}_n-\mathrm{N}_n\mathrm{Q}_n\}\end{array}\right]$$

$$=\sum_1^{\infty}\frac{\pi an^3p\sigma}{4\pi^2a^2p^2+\sigma^2n^2}(\mathrm{N}_n\mathrm{Q}_n-\mathrm{M}_n\mathrm{R}_n) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(vii.)}.$$

5. The couple vanishes when for all values of n,

$$\frac{\mathrm{M}_n}{\mathrm{Q}_n} = \frac{\mathrm{N}_n}{\mathrm{R}_n} = \tan\phi_n, \text{ say,}$$

so that H_0 is of the form

$$\Sigma\,[\mathrm{Q}_n\cos n\theta+\mathrm{R}_n\sin n\theta]\,\frac{\cos(pt-\phi_n)}{\cos\phi_n},$$

in other words, the couple vanishes when both parts of each harmonic are in the same phase, though that phase be not the same for all harmonics. As a particular case the remarkable result holds that whatever be the nature of the external field (it being made up, of course, of currents parallel to the axis), there will be no couple on the shell, provided the external field be altogether in the same phase.

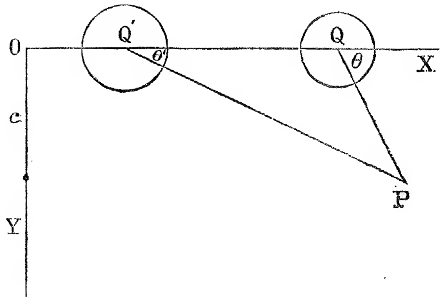
The Effect on an Infinite Cylindrical Shell, in the presence of an Alternating Current in a Parallel Wire, of the Interposition of a Parallel Cylindrical Shell.

6. Take the plane through the axes of the shells as that of ZOX, and a perpendicular plane through the wire as that of YOZ.

The shells are thin; let their radii be a, a', and their distances from the origin b and b'; let the distance of the wire from the origin be c, and the current in it $\mathrm{I} \cos pt$.

Let the position of *a pt.* P in space be determined by its distances r, r' from the axes of the shells and the angles θ, θ'.

Fig. 30.



Then if $b - b' \equiv d$ we have

$$\left.\begin{aligned} r \cos \theta &= r' \cos \theta' - d \\ r \sin \theta &= r' \sin \theta' \end{aligned}\right\}$$

Hence

$$re^{-i\theta} = r'e^{-i\theta'} - d\,;$$

therefore

$$\frac{e^{ni\theta}}{r^n} = \frac{1}{(-d + r'e^{-i\theta'})^n},$$

and if $r' < d$,

$$\frac{\cos n\theta}{r^n} = \frac{(-)^n}{d^n}\left[1 + \frac{nr'}{d}\cos\theta' + \frac{n.\overline{n+1}}{2!}\frac{r'^2}{d^2}\cos 2\theta' + \dots\right]$$

$$\frac{\sin n\theta}{r^n} = \frac{(-)^n}{d^n}\left[n\frac{n'}{d}\sin\theta' + \frac{n.\overline{n+1}}{2!}\frac{r'^2}{d^2}\sin 2\theta' + \dots\right].$$

So, too, if $r < d$,

$$r'e^{-i\theta} = re^{i\theta} + d,$$

and

$$\frac{\cos n\theta'}{r'^n} = \frac{1}{d^n}\left[1 + \frac{nr}{d}\cos\theta + \frac{n.\overline{n+1}}{2!}\frac{r^2}{d^2}\sin 2\theta + \dots\right]$$

$$\frac{\sin n\theta'}{r'^n} = \frac{1}{d^n}\left[n\frac{r}{d}\sin\theta + \frac{n.\overline{n+1}}{2!}\frac{r^2}{d^2}\sin 2\theta + \dots\right].$$

7. Due to the alternating current in the wire we have, over the a, b cylinder,

$$\mathrm{H}_0 = -\mathrm{I}\log\frac{(c - a\sin\theta)^2 + (b + a\cos\theta)^2}{\mathrm{D}^2}\cos pt$$

where D is a constant.

On putting

$$\left.\begin{aligned} b &= \rho \cos \gamma \\ c &= \rho \sin \gamma \end{aligned}\right\}$$

we get when $a < \rho$,

$$\begin{aligned} \mathrm{H}_0 &= - \mathrm{I} \, . \log \frac{\rho^2 + 2\rho a \cos \overline{\gamma + \theta} + a^2}{\mathrm{D}^2} \cos pt \\ &= - \cos pt \left[\log \frac{\rho^2}{\mathrm{D}^2} \right] \\ &\quad - 2\mathrm{I} \cos pt \left[\frac{a}{\rho} \cos \overline{\theta + \gamma} - \frac{a^2}{2\rho^2} \cos \overline{2\theta + \gamma} + \frac{a^3}{3\rho^3} \cos \overline{3\theta + \gamma} \right. \\ &\qquad \left. - \frac{a^4}{4\rho^4} \cos \overline{4\theta + \gamma} \ldots \right]. \end{aligned}$$

On substituting for

$$\rho \cos \theta, \quad \rho \sin 2\theta, \quad \text{\&c.},$$

we get to the fourth power of a/ρ,

$$\begin{aligned} \mathrm{H}_0 = + \mathrm{I} \cos pt \Bigg[& - \log \frac{b^2 + c^2}{\mathrm{D}^2} - \frac{2a}{b^2 + c^2} (b \cos \theta - c \sin \theta) \\ & + \frac{a^2}{(b^2 + c^2)^2} (\overline{- c^2 + b^2} \cos 2\theta - 2bc \sin 2\theta) \\ & + \frac{2a^2}{(b^2 + c^2)^2} (\overline{3bc^2 - b^3} \cos 3\theta - \overline{c^3 - 3b^2c} \sin 3\theta) \\ & + \frac{a^4}{2\,(b^2 + c^2)^4} (\overline{b^4 + c^4 - 6b^2c^2} \cos 4\theta - 4bc\, \overline{b^2 - c^2} \sin 4\theta) + \ldots \Bigg] \end{aligned}$$

8. The currents set up in both cylinders will clearly be parallel to their axes, and if those in the a, b shell produce at points on itself the momentum

$$\begin{aligned} \mathrm{H} = {} & \mathrm{M}_0 \sin pt + \mathrm{Q}_0 \cos pt \\ & + (\mathrm{M}_1 \cos \theta + \mathrm{N}_1 \sin \theta) \sin pt + (\mathrm{Q}_1 \cos \theta + \mathrm{R}_1 \sin \theta) \cos pt \\ & + (\mathrm{M}_2 \cos 2\theta + \mathrm{N}_2 \sin 2\theta) \sin pt + (\mathrm{Q}_2 \cos 2\theta + \mathrm{R}_2 \sin 2\theta) \cos pt \\ & + \text{higher harmonics,} \end{aligned}$$

and if the currents in the a', b' shell produce on itself a momentum, distinguished from the above by the dashing of the letters

$$\begin{aligned} \mathrm{H} = {} & \mathrm{M'}_0 \sin pt + \mathrm{Q'}_0 \cos pt \\ & + (\mathrm{M'}_1 \cos \theta + \ldots) \sin pt + (\ldots) \cos pt \\ & + (\mathrm{M'}_2 \cos 2\theta + \ldots) \sin pt + \ldots \\ & + \text{higher terms,} \end{aligned}$$

then in all we have, that on the a, b cylindrical surface, the momentum of the field that produces currents in that shell, is (to the 4th power of a, a', as will afterwards be explained)

$$
\begin{aligned}
&- \mathrm{I} \log \frac{b^2 + c^2}{\mathrm{D}^2} \cos pt \\
&- \mathrm{I} \frac{2a}{b^2 + c^2} (b \cos \theta - c \sin \theta) \cos pt \\
&+ \mathrm{I} \frac{a^2}{(b^2 + c^2)^2} (\overline{b^2 - c^2} \cos 2\theta - 2bc \sin 2\theta) \cos pt \\
&+ \mathrm{I} \frac{2a^3}{(b^2 + c^2)^3} (\overline{3bc^2 - b^3} \cos 3\theta - \overline{c^3 - 3b^2c} \sin 3\theta) \cos pt \\
&+ \mathrm{I} \frac{a^4}{2(b^2 + c^2)^4} (\overline{b^4 + c^4 - 6b^2c^2} \cos 4\theta + 4bc\,\overline{c^2 - b^2} \sin 4\theta) \cos pt \\
&+ \text{5th powers} \ldots \\
&+ \mathrm{M}'_0 \sin pt + \mathrm{Q}'_0 \cos pt \\
&+ (\mathrm{M}'_1 \sin pt + \mathrm{Q}'_1 \cos pt) \frac{a'}{d} \left(1 + \frac{a}{d} \cos \theta + \frac{a^2}{d^2} \cos 2\theta \ldots\right) \\
&+ (\mathrm{N}'_1 \sin pt + \mathrm{R}_1 \cos pt) \left(\frac{a'}{d}\right) \left(\frac{a}{d} \sin \theta + \frac{a^2}{d^2} \sin 2\theta \ldots\right) \\
&+ (\mathrm{M}'_2 \sin pt + \mathrm{Q}_2 \cos pt) \frac{a'^2}{d^2} (1 + \ldots) \\
&+ \text{5th and higher powers.}
\end{aligned}
$$

9. But if $\mathrm{D}_n \equiv 4\pi^2 a^2 p^2 + \sigma^2 n^2$, an external field of momentum

$$\mathrm{H}_0 = ({}^{`}\mathrm{M}_n \cos n\theta + {}^{`}\mathrm{N}_n \sin n\theta) \sin pt + ({}^{`}\mathrm{Q}_n \cos n\theta + {}^{`}\mathrm{R}_n \sin n\theta) \cos pt$$

at the surface of the a, b shell will produce in it currents whose momentum at the surface is by (v.)

$$
\begin{aligned}
\mathrm{H} = &- \frac{2\pi ap}{\mathrm{D}_n} [2\pi ap \sin pt + \sigma n \cos pt] [{}^{`}\mathrm{M}_n \cos n\theta + {}^{`}\mathrm{N}_n \sin n\theta] \\
&- \frac{2\pi ap}{\mathrm{D}_n} [2\pi ap \cos pt - \sigma n \sin pt] [{}^{`}\mathrm{Q}_n \cos n\theta + {}^{`}\mathrm{R}_n \sin n\theta].
\end{aligned}
$$

Applying this to the values of H and H_0 that we have recently found, we see that to the 4th powers of a. a' (as we shall shortly explain)

$$M_1 = \frac{2\pi a p \sigma}{D_1}\left[-\frac{2Iab}{b^2+c^2}+\frac{aa'}{d^2}Q'_1\right]-\frac{4\pi^2a^2p^2}{D_1}\frac{aa'}{d^2}M'_1,$$

$$N_1 = \frac{2\pi a p \sigma}{D_1}\left[+\frac{2Iac}{b^2+c^2}+\frac{aa'}{d^2}R'_1\right]-\frac{4\pi^2a^2p^2}{D_1}\frac{aa'}{d^2}N'_1,$$

$$Q_1 = -\frac{4\pi^2a^2p^2}{D_1}\left[-\frac{2Iab}{b^2+c^2}+\frac{aa'}{d^2}Q'_1\right]-\frac{2\pi a p \sigma}{D_1}\frac{aa'}{d^2}M'_1,$$

$$R_1 = -\frac{4\pi^2a^2p^2}{D_1}\left[+\frac{2Iac}{b^2+c^2}+\frac{aa'}{d^2}R'_1\right]-\frac{2\pi a p \sigma}{D_1}\frac{aa'}{d^2}N'_1,$$

$$M_2 = \frac{4\pi a p \sigma}{D_2}\left[\frac{a^2.\overline{b^2-c^2}}{(b^2+c^2)^2}I+\frac{a^2a'}{d^3}Q'_1\right]-\frac{4\pi^2a^2p^2}{D_2}\frac{a^2a'}{d^3}M'_1,$$

$$N_2 = \frac{4\pi a p \sigma}{D_2}\left[-\frac{2a^2bc}{(b^2+c^2)^2}I+\frac{a^2a'}{d^3}R'_1\right]-\frac{4\pi^2a^2p^2}{D_2}\frac{a^2a'}{d^3}N'_1,$$

$$Q_2 = -\frac{4\pi^2a^2p^2}{D_2}\left[\frac{a^2.\overline{b^2-c^2}}{(b^2+c^2)^2}I+\frac{a^2a'}{d^8}Q'_1\right]-\frac{4\pi a p \sigma}{D_2}\frac{a^2a'}{d^3}M'_1,$$

$$R_2 = \frac{4\pi^2a^2p^2}{D_2}\left[-\frac{2a^2bc}{(b^2+c^2)^2}I-\frac{a^2a'}{d^3}R'_1\right]-\frac{4\pi a p \sigma}{D_2}\frac{a^2a'}{d^3}N'_1,$$

$$M_3 = \frac{6\pi a p \sigma}{D_3}\frac{2a^3b\,(3c^2-b^2)}{(b^2+c^2)^3}I,$$

$$N_3 = -\frac{6\pi a p \sigma}{D_3}\frac{2a^3c\,(c^2-3b^2)}{(b^2+c)^3}I,$$

$$Q_3 = -\frac{4\pi^2a^2p^2}{D_3}\frac{2a^3b\,(3c^2-b^2}{(b^2+c\ )^3}I,$$

$$R_3 = \frac{4\pi^2a^2p^2}{D_3}\frac{2a^3c\,(c^2-3b^2)}{(b^2+c^2)^3}I,$$

$$M_4 = \frac{8\pi a p \sigma}{D_4}\frac{a^4\,(b^4+c^4-6b^2c^2)}{2\,(b^2+c^2)^4}I,$$

$$N_4 = -\frac{8\pi a p \sigma}{D_4}\frac{2a^4bc\,(b^2-c^2)}{(b^2+c^2)^4}I,$$

$$Q_4 = -\frac{4\pi^2a^2p^2}{D_4}\frac{a^4\,(b^4+c^4-6b^2c^2)}{2\,(b^2+c^2)^4}I,$$

$$R_4 = \frac{4\pi^2a^2p^2}{D_4}\frac{2a^4bc\,(b^2-c^2)}{(b^2+c^2)^4}I.$$

The equations that hold for the a', b' shell may be obtained from the above by interchanging dashed and undashed letters and altering the sign of d.

10. In order to justify the approximation we observe :—

(α). For ordinary values of the letters, $2\pi ap$ is comparable with σn ; in the case, for instance, of a copper shell of diameter 10 cm., and thickness $\frac{1}{3}$ cm.,

$$a = 5,$$

$$\sigma = 3 \times 1640 \text{ about} ;$$

and if

$$2\pi ap = \sigma n,$$

then $p = 160n$ about, which means that the number of reversals per second is about $55n$.

Hence, as far as we are concerned, we regard $4\pi ap\sigma/D_1$, or $8\pi^2a^2p^2/D_2$ as of *no* degree in a.

(β). It only remains to remark that the degree in a, a' of the most important term of each of the coefficients N_1, Q_3, &c, is that of their suffix 1, 3, &c.

11. We have

$$M'_1 = -\frac{4\pi a'p\sigma'}{D'_1}\frac{a'b'}{(b'^2+c^2)}\,I + \text{cubes}.$$

$$N'_1 = +\frac{4\pi a'p\sigma'}{D'_1}\frac{a'c}{b'^2+c^2}\,I + \text{cubes}.$$

$$Q'_1 = +\frac{8\pi^2a'^2p^2}{D'_1}\frac{a'b'}{b'^2+c^2}\,I + \ldots$$

$$R'_1 = -\frac{8\pi^2a'^2p^2}{D'_1}\frac{a'c}{b'^2+c^2}\,I + \ldots$$

Hence, as far as the fourth degree, in a, a',

$$\left.\begin{aligned} M_1 &= \frac{2\pi ap\sigma}{D_1}\left[-\frac{2Iab}{b^2+c^2}+\frac{8\pi^2a'^2p^2}{D'_1d^2}\frac{aa'^2b'}{b'^2+c^2}\,I\right]+\frac{16\pi^3a^2a'p^3\sigma'}{D_1\,D'_1\,d^2}\frac{aa'^2b'}{b'^2+c^2}\,I \\ N_1 &= \frac{2\pi ap\sigma}{D_1}\left[+\frac{2Iac}{b^2+c^2}-\frac{8\pi^2a'^2p^2}{D'_1}\frac{aa'^2c}{d^2(b'^2+c^2)}\,I\right]-\frac{16\pi^3a^2a'p^3\sigma'}{D_1\,D'_1}\frac{aa'^2c}{d^2(b'^2+c^2)}\,I \end{aligned}\right\} \text{(viii.)},$$

with similar values for Q_1 and R_1.

$$M_2 = I\,.\,\frac{4\pi ap\sigma}{D_2}\left[\frac{a^2(b^2-c^2)}{(b^2+c^2)^2}+\frac{8\pi^2a'^2p^2}{D'_1}\frac{a^2a'^2b'}{d^3(b'^2+c^2)}\right]+\frac{16\pi^3a^2a'p'\sigma'}{D'_1\,D_2}\frac{a^2a'^2b'}{d^3(b'^2+c^2)}\,I,$$

with similar values for N_2, Q_2, and R_2.

The values of the coefficients may be determined by the same method to a further degree of accuracy, and since the current w is given when the H due to it is given, *i.e.*,

$$w = \sum_{1}^{\infty} \frac{n}{2\pi a} \left[(\mathrm{M}_n \cos n\theta + \mathrm{N}_n \sin n\theta) \sin pt + (\mathrm{Q}_n \cos n\theta + \mathrm{R}_n \sin n\theta) \cos pt\right],$$

thus, the currents may be determined to any degree of closeness.

12. In order to find the couple that acts on the a, b shell, we use our previous result (vii.),

$$\Sigma \frac{\pi a p \sigma n^3}{\mathrm{D}_n} \left[{}^{\backprime}\mathrm{Q}_n {}^{\backprime}\mathrm{N}_n - {}^{\backprime}\mathrm{M}_n {}^{\backprime}\mathrm{R}_n\right].$$

As far as terms of the sixth degree in a, a' this is (writing $b^2 + c^2 \equiv \rho^2$, $b'^2 + c^2 \equiv \rho'^2$)

$$\frac{\pi a p \sigma}{\mathrm{D}_1}\left[\left(-\frac{2ab}{\rho^2}\mathrm{I} + \frac{aa'}{d^2}\mathrm{Q}'_1\right)\frac{aa'}{d^2}\mathrm{N}'_1 - \frac{aa'}{d^2}\mathrm{M}'_1\left(+\frac{2ac}{\rho^2}\mathrm{I} + \frac{aa'}{d^2}\mathrm{R}'_1\right)\right]$$
$$+ \frac{8\pi a p \sigma}{\mathrm{D}_2}\left[\frac{a^2\,.\,\overline{b^2 - c^2}}{\rho^4}\mathrm{I}\,.\,\frac{a^2 a'}{d^3}\mathrm{N}'_1 + \frac{a^2 a'}{d^3}\mathrm{M}'_1\,.\,\frac{2bca^2}{\rho^4}\mathrm{I}\right],$$

remembering that the third harmonic terms are (to this order) in the same phase and give no couple (${}^{\backprime}\mathrm{M}_3 = 0$, ${}^{\backprime}\mathrm{N}_3 = 0$).

This expression is

$$\frac{\pi a p \sigma}{\mathrm{D}_1}\,.\,\frac{aa'}{d^2}\left[-\left(\frac{2ab}{\rho^2}\mathrm{N}'_1 + \frac{2ac}{\rho^2}\mathrm{M}'_1\right)\mathrm{I} + \frac{aa'}{d^2}(\mathrm{Q}'_1\mathrm{N}'_1 - \mathrm{M}'_1\mathrm{R}'_1)\right]$$
$$+ \frac{8\pi a p \sigma}{\mathrm{D}_2}\frac{a^2 a'}{d^3}\frac{a^2\mathrm{I}}{\rho^4}\left[(b^2 - c^2)\,\mathrm{N}'_1 + 2bc\,\mathrm{M}'_1\right].$$

Now, $\mathrm{Q}'_1\mathrm{N}'_1 - \mathrm{M}'_1\mathrm{R}'_1$ has to be calculated to the second order only, and on reference to the values of these coordinates to the first order it will be seen that it is zero. Within the first bracket M'_1 and N'_1 are required to the third degree. We have (viii.)

$$\mathrm{M}'_1 = -\frac{2\pi a' p \sigma'}{\mathrm{D}'_1}\frac{2\mathrm{I}a'b'}{\rho'^2} + \frac{16\pi^3 a^3 a'^2 p^3 b}{\mathrm{D}_1\mathrm{D}'_1 d^2\rho^2}\mathrm{I}\,(a'\sigma + a\sigma'),$$
$$\mathrm{N}'_1 = +\frac{2\pi a' p \sigma'}{\mathrm{D}'_1}\frac{2\mathrm{I}a'c}{\rho'^2} - \frac{16\pi^3 a^3 a'^2 p^3 c}{\mathrm{D}_1\mathrm{D}'_1 d^2\rho^2}\mathrm{I}\,(a'\sigma + a\sigma').$$

Hence, for the couple we get

$$\frac{\pi a p \sigma}{\mathrm{D}_1}\frac{aa'}{d^2}\frac{2a}{\rho^2}\mathrm{I}^2\left[\frac{2\pi a' p \sigma'}{\mathrm{D}'_1}\frac{2a'}{\rho'^2}(b'c - bc) + \frac{16\pi^3 a^3 a'^2 p^3}{\mathrm{D}_1\mathrm{D}'_1 d^2\rho^2}(a'\sigma + a\sigma')(bc - bc)\right]$$
$$+ \frac{8\pi a p \sigma}{\mathrm{D}_2}\frac{a^2 a'}{d^3}\frac{a^2\mathrm{I}}{\rho^4}\frac{2\pi a' p \sigma'}{\mathrm{D}'_1}\frac{2\mathrm{I}a'}{\rho'^2}\left[-2bc\,.\,b' + (b^2 - c^2)\,c\right].$$

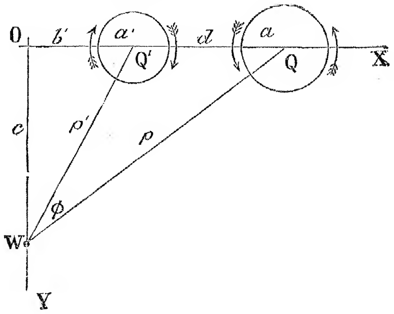
This consolidates finally into

$$-\mathrm{I}^2\,.\,\frac{8\pi^2 p^2 \sigma\sigma' c a^3 a'^3}{\rho^2\rho'^2 d}\left[\frac{1}{\mathrm{D}_1\mathrm{D}_1'} + \frac{4a^2(\rho'^2 - d^2)}{\mathrm{D}'_1\mathrm{D}_2\rho^2 d^2}\right] \quad . \quad . \quad . \quad . \quad \text{(ix)}.$$

13. It will be noticed that the more important term is symmetrical, except that it

changes sign if it be applied to the a', b' shell instead of the a, b; the direction of rotation is negative when $b - b'$ is positive, *i.e.*, is as indicated in the figure, and is *à priori* likely, since the repulsion is likelier to be stronger where the currents and field are stronger, *i.e.*, on the side nearest the wire of the shell to the left and on the unshielded side of that to the right. The couple changes sign with c as it should, and vanishes if σ or σ' vanish. The term of the eighth degree vanishes when $\rho' = d$, *i.e.*, $QQ' = Q'W$.

Fig. 31.



The Effects on two parallel Thin Infinite Circular Cylindrical Shells of a Thin Filament parallel to them, and alternately Magnetised in a direction perpendicular to its Length.

14. Let the same axes and coordinates be taken as before; the filament may be regarded as an electromagnet consisting of a current

$$\frac{A}{\kappa} \cos pt \quad \text{(where } \kappa \text{ is small)},$$

parallel to OZ at $x = 0,\ y = c$, and a current

$$-\frac{A}{\kappa} \cos pt,$$

parallel to OZ at $x = -\kappa \sin \alpha,\ y = c - \kappa \cos \alpha$.

The direction of magnetisation will then make an angle of α with OX, and the strength will be $A \cos pt$ per unit length.

If $I \, . f(b, b', c, d)$ were a coefficient in the expansion of the electromagnetic momentum in the case of a current $I \cos pt$ at $x = 0,\ y = c$, then, with the electromagnet, that coefficient will be the limit of

$$\frac{A}{\kappa} f(b, b', c, d) - \frac{A}{\kappa} f(b + \kappa \sin \alpha,\ b' + \kappa \sin \alpha,\ c - \kappa \cos \alpha,\ d),$$

i.e.,

$$A \left[- \sin \alpha \, . \left(\frac{\partial f}{\partial b} + \frac{\partial f}{\partial b'} \right) + \cos \alpha \frac{\partial f}{\partial c} \right],$$

but the same plan will not avail when applied to find the new couple (by directly differentiating the couple due to the current I cos pt), owing to the interaction of the currents set up by the two elementary currents of the electromagnet.

We shall have to use our previous result, that if the external field on the surface of a shell has momentum

$$\Sigma\left[({}^{\backprime}\mathrm{M}_n \cos n\theta + {}^{\backprime}\mathrm{N}_n \sin n\theta) \sin pt + ({}^{\backprime}\mathrm{Q}_n \cos n\theta + {}^{\backprime}\mathrm{R}_n \sin n\theta) \cos pt\right],$$

the couple is by (vii.),

$$\Sigma \frac{\pi a p \sigma n^3}{\mathrm{D}_n} \left[{}^{\backprime}\mathrm{N}_n {}^{\backprime}\mathrm{Q}_n - {}^{\backprime}\mathrm{M}_n {}^{\backprime}\mathrm{R}_n\right].$$

On reference it will be seen that due to the current I cos pt, we had

$$\begin{aligned}
{}^{\backprime}\mathrm{M}_1 &= -\frac{aa'}{d^2}\frac{4\pi a' p \sigma'}{\mathrm{D}'_1}\frac{a'b'}{\rho'^2}\,\mathrm{I} + \text{cubes.} \\
{}^{\backprime}\mathrm{N} &= +\frac{aa'}{d^2}\frac{4\pi a' p \sigma'}{\mathrm{D}'_1}\frac{a'c}{\rho'^2}\,\mathrm{I}\ldots \\
{}^{\backprime}\mathrm{Q}_1 &= -\frac{2ab}{\rho^2}\,\mathrm{I} + \frac{aa'}{d^2}\frac{8\pi^2 a'^2 p^2}{\mathrm{D}'_1}\frac{a'b'}{\rho'^2}\,\mathrm{I}\ldots \\
{}^{\backprime}\mathrm{R}_1 &= +\frac{2ac}{\rho^2}\,\mathrm{I} - \frac{aa'}{d^2}\frac{8\pi^2 a'^2 p^2}{\mathrm{D}'_1}\frac{a'c}{\rho'^2}\,\mathrm{I}\ldots
\end{aligned}$$

Now, with the electromagnet instead of the single current (remembering $\rho \cos \gamma = b$, $\rho \sin \gamma = c$), $\frac{b}{\rho^2}$ I will become

$$+\mathrm{A}\left[-\frac{\sin \alpha}{\rho^2} + \frac{2b}{\rho^2}(+b \sin \alpha + c \cos \alpha)\right]$$

or

$$\mathrm{A}\left[-\frac{\sin \alpha}{\rho^2} + \frac{2 \cos \gamma}{\rho^2} \sin \overline{\alpha + \gamma}\right].$$

Also $\frac{c}{\rho^2}$ I will become

$$\frac{\mathrm{A}}{\rho^2}\left[\cos \alpha - 2 \sin \gamma \sin \overline{\alpha + \gamma}\right]$$

and for $\frac{b'}{\rho'^2}$ I, $\frac{c}{\rho'^2}$ I, we have merely to add dashes to ρ and γ.

If we omit terms of the *eighth* degree in a_1 a', the value of the couple will be

$$\frac{\pi a p \sigma}{\mathrm{D}_1}\left[{}^{\backprime\backprime}\mathrm{N}_1 {}^{\backprime\backprime}\mathrm{Q}_1 - {}^{\backprime\backprime}\mathrm{M}_1 {}^{\backprime\backprime}\mathrm{R}_1\right],$$

where

$$\text{``}M_1 = -\frac{4\pi p\sigma' a a'^3}{d^2 D'_1 \rho'^2} A\,(-\sin\alpha + 2\cos\gamma'\sin\overline{\alpha+\gamma'}),$$

$$\text{``}N_1 = -\frac{4\pi p\sigma' a a'^3}{d^2 D'_1 \rho'^3} A\,(\cos\alpha - 2\sin\gamma'\sin\overline{\alpha+\gamma'}),$$

$$\text{``}Q_1 = -\frac{2aA}{\rho^2}(-\sin\alpha + 2\cos\gamma\sin\overline{\alpha+\gamma}) + \frac{8\pi^2 a a'^4 p^2}{d^2 D'_1 \rho'^2} A\,(-\sin\alpha + 2\cos\gamma'\sin\overline{\alpha+\gamma'}),$$

$$\text{``}R_1 = -\frac{2aA}{\rho^2}(\cos\alpha - 2\sin\gamma\sin\overline{\alpha+\gamma}) + \frac{8\pi^2 a a'^4 p^2}{d^2 D'_1 \rho'^2} A\,(\cos\alpha - 2\sin\gamma'\sin\overline{\alpha+\gamma'}).$$

On multiplying up, we notice that the second terms of $\text{``}Q_1$ and $\text{``}R_1$ cancel, when multiplied respectively by $\text{``}N_1$ and $\text{``}M_1$, and we get for the couple

$$\begin{aligned} &\frac{\pi a p \sigma}{D_1}\,\frac{2aA}{\rho^2}\,\frac{4\pi p\sigma' a a'^3}{d^2 D'_1 \rho'^2} A \\ &\qquad \times [(\cos\alpha - 2\sin\gamma'\sin\overline{\alpha+\gamma'})(-\sin\alpha + 2\cos\gamma\sin\overline{\alpha+\gamma}) \\ &\qquad\qquad - (\cos\alpha - 2\sin\gamma\sin\overline{\alpha+\gamma})(-\sin\alpha + 2\cos\gamma'\sin\overline{\alpha+\gamma'})] \end{aligned}$$

or

$$\begin{aligned} A^2\frac{8\pi^2 p^2 \sigma\sigma' a^3 a'^3}{d^2 D_1 D'_1 \rho^2 \rho'^2} &[2\sin\overline{\alpha+\gamma}\,(\cos\gamma\cos\alpha - \sin\gamma\sin\alpha) \\ &- 2\sin\overline{\alpha+\gamma'}\,(\cos\gamma'\cos\alpha - \sin\gamma'\sin\alpha) \\ &- 4\sin\overline{\alpha+\gamma}\sin\overline{\alpha+\gamma'}\,(\cos\gamma\sin\gamma' - \cos\gamma'\sin\gamma)]. \end{aligned}$$

The expression within the bracket is

$$\sin 2\overline{\alpha+\gamma} - \sin 2\overline{\alpha+\gamma'} - 4\sin\overline{\alpha+\gamma}\sin\overline{\alpha+\gamma'}\sin\overline{\gamma'-\gamma}$$

or

$$-2\sin\overline{\gamma'-\gamma}\,[\cos\overline{2\alpha+\gamma+\gamma'} + 2\sin\overline{\alpha+\gamma}\sin\overline{\alpha+\gamma'}]$$

or

$$-2\sin\overline{\gamma'-\gamma}\cos\overline{\gamma'-\gamma}.$$

Hence, omitting terms of the *eighth* degree, the couple on the a, b shell is

$$-A^2\frac{8\pi^2 p^2\sigma\sigma' a^3 a'^3}{d^2 D_1 D'_1 \rho^2 \rho'^2}\sin 2\phi,$$

where ϕ is $(\gamma' - \gamma)$, the angle subtended by the axes of the shells at the electromagnetic filament.

15. To this order then we have the following results :—

(α.) The couple exerted on a shell is independent of α, *i.e.*, of the direction of the axis of the electromagnet.

(β.) The shells have equal couples in opposite directions, the parts of the shells directed towards one another being driven towards the electromagnet if ϕ is less than a right angle and driven away from it if ϕ exceeds a right angle. If ϕ be a right angle the couple vanishes.

Fig. 32.

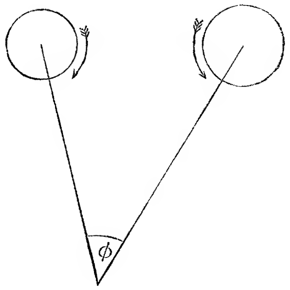
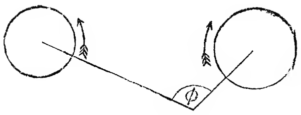


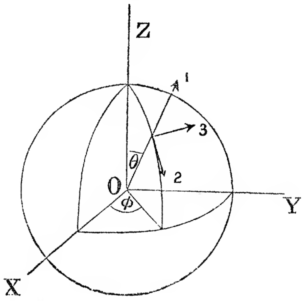
Fig. 33.



The Mechanical Effect of Currents set up in a Thin Spherical Shell.

16. In obtaining the currents set up by a given field we follow Professor C. Niven: f, using spherical coordinates and the ordinary notation, Φ be the current function of the distribution in the shell, P the potential of imaginary matter spread over it with surface density Φ, then

Fig. 34.



$$\Omega = -\frac{1}{a}\frac{d}{dr}(\mathrm{P}r), \quad \mathrm{F} = 0,$$

$$\mathrm{G} = \frac{1}{a}\frac{d\mathrm{P}}{\sin\theta\, d\phi}, \quad \mathrm{H} = -\frac{d\mathrm{P}}{a\, d\theta}.$$

The components of the current are

$$u = 0, \quad v = -\frac{\partial\Phi}{a\sin\theta\,\partial\phi}, \quad w = \frac{\partial\Phi}{a\,\partial\theta}.$$

If the external field have magnetic potential Ω_0, and if

$$\Omega_0 = -\frac{1}{a}\frac{d}{dr}(\mathrm{P}_0 r)$$

(ever distinguishing quantities that refer to the externally applied system by a zero suffix), then the equations connecting the components of the vector potential with the components of induction are satisfied by

$$\mathrm{F}_0 = 0, \quad \mathrm{G}_0 = \frac{1}{a}\frac{\partial \mathrm{P}_0}{\sin\theta\,\partial\phi}, \quad \mathrm{H}_0 = -\frac{\partial \mathrm{P}_0}{a\,\partial\theta}.$$

The equations giving the currents are

$$-\sigma\frac{\partial\Phi}{a\sin\theta\,\partial\phi} = -\frac{d}{dt}\left[\frac{1}{a\sin\theta}\frac{\partial}{\partial\phi}(\mathrm{P}+\mathrm{P}_0)\right] - \frac{\partial\psi}{a\,\partial\theta},$$

$$+\sigma\frac{\partial\Phi}{a\,\partial\theta} = -\frac{d}{dt}\left[-\frac{1}{a}\frac{\partial}{\partial\theta}(\mathrm{P}+\mathrm{P}_0)\right] - \frac{\partial\psi}{a\sin\theta\,\partial\phi},$$

and are satisfied by

$$\psi = 0,$$

$$\sigma\Phi = \frac{d}{dt}(\mathrm{P}+\mathrm{P}_0).$$

Let us consider the case

$$\Omega_0 = \mathrm{A}\left(\frac{r}{a}\right)^n \mathrm{Y}_n e^{ipt},$$

where a is the radius of the shell, and Y_n is a spherical harmonic of degree n.

We have

$$\mathrm{P}_0 = -\mathrm{A}\frac{r^n}{a^{n-1}}\frac{\mathrm{Y}_n}{n+1}e^{ipt} = -\mathrm{A}\frac{a}{n+1}\mathrm{Y}_n e^{ipt} \text{ at the surface.}$$

If due to this

$$\Phi = \mathrm{B}\mathrm{Y}_n e^{ipt};$$

then

$$\mathrm{P} = \frac{4\pi a}{2n+1}\mathrm{B}\mathrm{Y}_n e^{ipt}$$

at the surface, and the equation for B is

$$\sigma\mathrm{B} = ip\left[\frac{4\pi a}{2n+1}\mathrm{B} - \frac{a}{1+n}\mathrm{A}\right],$$

therefore

$$\mathrm{B} = \frac{-\mathrm{A}\dfrac{iap}{n+1}}{\dfrac{4i\pi pa}{2n+1}+\sigma}$$

$$= \mathrm{A}\frac{\overline{2n+1}\,ap}{n+1}\,\frac{4\pi pa - i\,\overline{2n+1}\,\sigma}{16\pi^2a^2p^2+\overline{2n+1}^2\sigma^2}.$$

Denoting then

$$16\pi^2a^2p^2+\overline{2n+1}^2\sigma^2$$

by Δ_n, we shall have corresponding to

$$\Omega_0 = \mathrm{AY}_n \cos pt$$

at the surface, the value

$$\Phi = \mathrm{A}\frac{\overline{2n+1}\,pa}{(n+1)\,\Delta_n}\mathrm{Y}_n\,(4\pi pa\cos pt + \overline{2n+1}\,\sigma\sin pt),$$

and corresponding to

$$\Omega_0 = \mathrm{BZ}_n \sin pt$$

$$\Phi = \mathrm{BZ}_n\frac{\overline{2n+1}\,pa}{\overline{n+1}\,\Delta_n}\,(4\pi pa\sin pt - \overline{2n+1}\,\sigma\cos pt.)$$

17. If the components of magnetic induction or magnetic force along the directions

$$\delta r, \qquad r\,\delta\theta, \qquad r\sin\theta\,\delta\phi$$

be α, β, γ, the components of electromagnetic force in these directions will be

$$\gamma v - \beta w, \quad \text{or} \quad -\gamma\frac{\partial\Phi}{a\sin\theta\,\partial\phi} - \beta\frac{\partial\Phi}{a\,\partial\theta}$$

$$\alpha w, \quad \text{or} \quad \alpha\frac{\partial\Phi}{a\,\partial\theta}$$

$$-\alpha v, \quad \text{or} \quad \alpha\frac{\partial\Phi}{a\sin\theta\,\partial\phi}.$$

The couple about the axis of z will be the integral over the surface of

$$-\alpha v\,.\,a\sin\theta,$$

where in α it will be sufficient to take into account the external field (as in the case of the cylinder), for the shell will not exert any couple on itself. This couple is

$$\iint d\mathrm{S}\,.\,\alpha\frac{\partial\Phi}{\partial\phi}\,.$$

Now if Ω_0 be periodic in the time $2\pi/p$, it can be expanded near to the surface of the sphere in the series

$$\sum_{n=1}^{n=\infty}\left(\frac{r}{a}\right)^n [\mathrm{Y}_n \cos pt + \mathrm{Z}_n \sin pt]$$

where Y_n, Z_n include only harmonics of the n^{th} degree.

Due to this value of Ω_0 we shall have at the surface

$$\alpha = -\Sigma \frac{n}{a}[\mathrm{Y}_n \cos pt + \mathrm{Z}_n \sin pt] \quad . \; . \; . \; . \; . \; . \; . \; . \quad (\text{x.}),$$

and

$$\Phi = \Sigma \frac{(2n+1)pa}{(n+1)\Delta_n}[4\pi pa\,(\mathrm{Y}_n \cos pt + \mathrm{Z}_n \sin pt) \\ + \overline{2n+1}\,\sigma\,(\mathrm{Y}_n \sin pt - \mathrm{Z}_n \cos pt)] \quad . \; . \; . \quad (\text{xi.}).$$

The mean value of $\sin^2 pt$ and $\cos^2 pt$ being $\frac{1}{2}$, and of $\sin pt \cos pt$ zero, that of

$$\iint d\mathrm{S} \,.\, \alpha \frac{\partial \Phi}{\partial \phi}$$

will be, denoting differentiation to ϕ by dashes,

$$\tfrac{1}{2}\iint d\mathrm{S}\left[-\Sigma\left(\frac{n}{a}\,\mathrm{Y}_n\right)\Sigma\frac{\overline{2n+1}\,pa}{\overline{n+1}\,\Delta_n}(4\pi pa\mathrm{Y}'_n - \overline{2n+1}\,\sigma\mathrm{Z}'_n) \\ - \Sigma\left(\frac{n}{a}\,\mathrm{Z}_n\right)\Sigma\frac{\overline{2n+1}\,pa}{\overline{n+1}\,\Delta_n}(4\pi pa\mathrm{Z}'_n + \overline{2n+1}\,\sigma\mathrm{Y}'_n)\right]$$

Now Y'_n, Z'_n are harmonics of the n^{th} degree and will give zero when multiplied by harmonics of other degrees than n^{th} and integrated over the sphere: hence it is sufficient to write for the couple

$$\tfrac{1}{2}\iint d\mathrm{S}\sum_{1}^{\infty}\frac{n\,.\,\overline{2n+1}\,p}{(n+1)\,\Delta_n}[(2n+1)\,\sigma\,(\mathrm{Y}_n\mathrm{Z}'_n - \mathrm{Y}'_n\mathrm{Z}_n) - 4\pi pa\,(\mathrm{Y}_n\mathrm{Y}'_n + \mathrm{Z}_n\mathrm{Z}'_n)].$$

Also $\mathrm{Y}_n\,(\partial\mathrm{Y}_n/\partial\phi)$ being a perfect differential, vanishes when integrated to ϕ from 0 to 2π: thus

$$\iint d\mathrm{S}\,.\,\mathrm{Y}_n\mathrm{Y}'_n = 0,$$

$$\iint d\mathrm{S}\,.\,\mathrm{Z}_n\mathrm{Z}'_n = 0,$$

and the couple is

$$\sum_{1}^{\infty}\frac{n\,.\,(2n+1)^2\,p\sigma}{2\,(n+1)\,\Delta_n}\iint d\mathrm{S}\,.\left[\mathrm{Y}_n\frac{\partial\mathrm{Z}_n}{\partial\phi} - \mathrm{Z}_n\frac{\partial\mathrm{Y}_n}{\partial\phi}\right] \quad . \; . \; . \; . \; . \; . \quad (\text{xii.}).$$

[It might be objected that being given the external field, we have no right to take $F_0 = 0$ at the surface; we have proved, however, in our introductory work, that if Ω_0 be given, an alteration in the normal component of the vector potential only introduces an alteration in the electrostatic potential.]

18. Hence it follows that—

(α.) If the external field be symmetrical round the axis of Z, so that Ω is independent of ϕ, the couple will be zero;

(β.) if the external field be completely in the same phase, or if when expanded in harmonics over the surface its form is

$$\Omega_0 = \Sigma\, U_n \cos (pt + \epsilon_n),$$

where ϵ_n is independent of the coordinates θ, ϕ, then the couple vanishes.

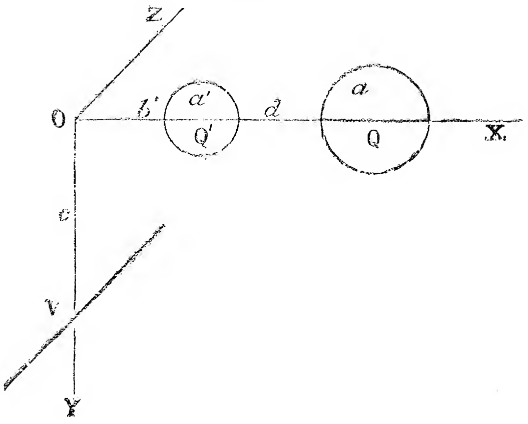
From this it follows that if we have an exciting field in the same phase, and introduce any number of perfectly conducting bodies, the couple will still be zero.

The Couples on two Thin Spherical Shells due to the presence of an Alternating Current in a Straight Infinite Wire.

19. Let the line joining the centres of the shells be taken as OX and the shortest distance between this line and the infinite wire as OY.

Let the radii of the shells be a, a', and their central distances from O, b and b': let the distance from O of the wire be c.

Fig. 35.



If a point in space have polar coordinates r, θ, ϕ when referred to Q as origin and axes parallel to OX, OY, OZ, and coordinates r', θ', ϕ' to Q' as origin, its Cartesian coordinates will be

$$\begin{aligned} x &= b + r \sin\theta \cos\phi \\ &= b' + r' \sin\theta' \cos\phi' \\ y &= r \sin\theta \sin\phi \\ &= r' \sin\theta' \sin\phi' \\ z &= r \cos\theta \\ &= r' \cos\theta'. \end{aligned}$$

Thus (if $b - b' \equiv d$)

$$\left.\begin{aligned} r' \sin\theta' \cos\phi' &= r \sin\theta \cos\phi + d \\ r' \sin\theta' \sin\phi' &= r \sin\theta \sin\phi \\ r' \cos\theta' &= r \cos\theta \end{aligned}\right\} \quad \ldots\ldots \quad \text{(xiii.)},$$

so that

$$r'^2 = r^2 + 2\,dr \sin\theta \cos\phi + d^2,$$

and when $r = a$ and is less than d,

$$\frac{1}{r'} = \frac{1}{d}\left[1 - \frac{a}{d}\sin\theta\cos\phi \ldots\right],$$

so that, omitting second harmonics,

$$\begin{aligned} \frac{\sin\theta'\cos\phi'}{r'^2} &= \frac{1}{d^3}\left(1 - \frac{3a}{d}\sin\theta\cos\phi\right)(d + a\sin\theta\cos\phi) \\ &= \frac{1}{d^2} - \frac{2a}{d^3}\sin\theta\cos\phi + \text{higher harmonics.} \end{aligned}$$

$$\frac{\sin\theta'\sin\phi'}{r'^2} = \frac{a\sin\theta\sin\phi}{d^3}\ \ldots$$

$$\frac{\cos\theta'}{r'^2} = \frac{a\cos\theta}{d^3}\ \ldots$$

20. Hence if the currents in the a, b surface produce upon that surface a magnetic potential

$$\begin{aligned} \Omega = {} & (\mathrm{A}\sin\theta\cos\phi + \mathrm{B}\sin\theta\sin\phi + \mathrm{C}\cos\theta)\cos pt \\ & + (\mathrm{D}\sin\theta\cos\phi + \mathrm{E}\sin\theta\sin\phi + \mathrm{F}\cos\theta)\sin pt \\ & + \text{harmonics of second and higher orders,} \end{aligned}$$

and if the currents in the a', b' shell produce upon itself a potential Ω', whose value is distinguished by dashes from the above, then the value of Ω' upon the a, b shell will be

$$
\begin{aligned}
&\frac{a'^2}{d^2}\left(\mathrm{A}'\cos pt + \mathrm{D}'\sin pt\right)\\
&+\frac{aa'^2}{d^3}\left[-2\mathrm{A}'\sin\theta\cos\phi + \mathrm{B}'\sin\theta\sin\phi + \mathrm{C}'\cos\theta\right]\cos pt\\
&+\frac{aa'^2}{d^3}\left[-2\mathrm{D}'\sin\theta\cos\phi + \mathrm{E}'\sin\theta\sin\phi + \mathrm{F}'\cos\theta\right]\sin pt\\
&+\text{higher harmonics.}
\end{aligned}
$$

The magnetic potential at x, y, z of the current in the infinite wire is

$$2\mathrm{I}\tan^{-1}\frac{c-y}{x}\cos pt,$$

and hence at θ, ϕ on the a, b sphere is

$$2\mathrm{I}\left[\tan^{-1}\frac{c - a\sin\theta\sin\phi}{b + a\sin\theta\cos\phi}\right]\cos pt,$$

or

$$2\mathrm{I}\cos pt\left[\tan^{-1}\frac{c}{b} - \frac{ab\sin\theta\sin\phi}{b^2+c^2} - \frac{ac\sin\theta\cos\phi}{b^2+c^2}\ldots\right],$$

omitting harmonics of second and higher degrees.

Hence the first harmonic of the magnetic potential of the system external to the a, b shell is upon its surface,

$$
\begin{aligned}
\Omega_0 = ({}^{\backprime}\mathrm{A}\sin\theta\cos\phi + {}^{\backprime}\mathrm{B}\sin\theta\sin\phi + {}^{\backprime}\mathrm{C}\cos\theta)\cos pt&\\
+ ({}^{\backprime}\mathrm{D}\sin\theta\cos\phi + {}^{\backprime}\mathrm{E}\sin\theta\sin\phi + {}^{\backprime}\mathrm{F}\cos\theta)\sin pt&,
\end{aligned}
$$

where, if $b^2 + c^2 = \rho^2$,

$$
\begin{aligned}
{}^{\backprime}\mathrm{A} &= -\frac{2\mathrm{I}a}{\rho^2} - \frac{2aa'^2}{d^3}\mathrm{A}'\\
{}^{\backprime}\mathrm{B} &= -\frac{2\mathrm{I}ab}{\rho^2} + \frac{aa'^2}{d^3}\mathrm{B}'\\
{}^{\backprime}\mathrm{D} &= \qquad\quad\; - \frac{2aa'^2}{d^3}\mathrm{D}'\\
{}^{\backprime}\mathrm{E} &= \qquad\quad\; + \frac{aa'^2}{d^3}\mathrm{E}'\\
{}^{\backprime}\mathrm{C} &= \qquad\quad\; + \frac{aa'^2}{d^3}\mathrm{C}'\\
{}^{\backprime}\mathrm{F} &= \qquad\quad\; + \frac{aa'^2}{d^3}\mathrm{F}'.
\end{aligned}
$$

Now we have shown (xi.) that a field

$$\Omega_0 = \mathrm{Y}_n e^{ipt}$$

produces a current function

$$\Phi = \frac{\overline{2n+1}\, ap}{(n+1)\,\Delta_n}\left(4\pi pa - i\,.\,\overline{2n+1}\,\sigma\right)\mathrm{Y}_n e^{ipt}.$$

The P of this is

$$\frac{4\pi a}{2n+1}\,.\,\Phi\,.\left(\frac{a}{r}\right)^{n+1},$$

and its magnetic potential at the surface

$$\frac{4\pi n}{(2n+1)}\,\Phi.$$

Hence the magnetic potential of the currents induced in the a, b shell by the system

$$\Omega_0 = ({}^{\backprime}\mathrm{A}\sin\theta\cos\phi + {}^{\backprime}\mathrm{B}\sin\theta\sin\phi + {}^{\backprime}\mathrm{C}\cos\theta)\,pt$$
$$+ ({}^{\backprime}\mathrm{D}\sin\theta\cos\phi + {}^{\backprime}\mathrm{E}\sin\theta\sin\phi + {}^{\backprime}\mathrm{F}\cos\theta)\sin pt$$

will be

$$\Omega = \frac{2\pi pa}{\Delta_1}\big[({}^{\backprime}\mathrm{A}\sin\theta\cos\phi + {}^{\backprime}\mathrm{B}\sin\theta\sin\phi + {}^{\backprime}\mathrm{C}\cos\theta)\,(4\pi pa\cos pt + 3\sigma\sin pt)$$
$$+ ({}^{\backprime}\mathrm{D}\sin\theta\cos\phi + {}^{\backprime}\mathrm{E}\sin\theta\sin\phi + {}^{\backprime}\mathrm{F}\cos\theta)\,(4\pi pa\sin pt - 3\sigma\cos pt)\big].$$

Comparing this with

$$(\mathrm{A}\sin\theta\cos\phi + \mathrm{B}\sin\theta\sin\phi + \mathrm{C}\cos\theta)\cos pt$$
$$+ (\mathrm{D}\sin\theta\cos\phi + \mathrm{E}\sin\theta\sin\phi + \mathrm{F}\cos\theta)\sin pt,$$

we have, on giving ${}^{\backprime}\mathrm{A}$. their values,

$$\mathrm{A} = \frac{2\pi pa}{\Delta_1}\left[4\pi pa\left(-\frac{2\mathrm{I}ac}{\rho^2} - \frac{2aa'^2}{d^3}\,\mathrm{A}'\right) - 3\sigma\left(-\frac{2aa'^2}{d^3}\,\mathrm{D}'\right)\right],$$

$$\mathrm{B} = \frac{2\pi pa}{\Delta_1}\left[4\pi pa\left(-\frac{2\mathrm{I}ab}{\rho^2} + \frac{aa'^2}{d^3}\,\mathrm{B}'\right) - 3\sigma\left(\frac{aa'^2}{d^3}\,\mathrm{E}'\right)\right],$$

$$\mathrm{C} = \frac{2\pi pa}{\Delta_1}\left[4\pi pa\left(\frac{aa'^2}{d^3}\,\mathrm{C}'\right) - 3\sigma\,\frac{aa'^2}{d^3}\,\mathrm{F}'\right],$$

$$\mathrm{D} = \frac{2\pi pa}{\Delta_1}\left[3\sigma\left(-\frac{2\mathrm{I}ac}{\rho^2} - \frac{2aa'^2}{d^3}\,\mathrm{A}'\right) + 4\pi pa\left(-\frac{2aa'^2}{d^3}\,\mathrm{D}'\right)\right],$$

$$\mathrm{E} = \frac{2\pi pa}{\Delta_1}\left[3\sigma\left(-\frac{2\mathrm{I}ab}{\rho^2} + \frac{aa'^2}{d^3}\,\mathrm{B}'\right) + 4\pi pa\left(\frac{aa'^2}{d^3}\,\mathrm{E}'\right)\right],$$

$$\mathrm{F} = \frac{2\pi pa}{\Delta_1}\left[3\sigma\left(\frac{aa'^2}{d^3}\,\mathrm{C}'\right) + 4\pi pa\,\frac{aa'^2}{d^3}\,\mathrm{F}'\right].$$

21. There are, of course, six similar equations obtained from consideration of the currents in the a', b' shell: these equations are accurate, since the introduction of further approximations gives rise to harmonics of higher orders than the first.

From the four equations giving C, F, C$'$ and F$'$, it follows that all these quantities are accurately zero, as might have been expected, since the system is unaltered on taking $-z$ for $+z$, *i.e.*, putting $(\pi - \theta)$ for θ.

From these equations it follows, as in the case of the cylinder, that the principal terms of the first harmonics are of the first degree in the radii, but their next are of the fourth (not the third, as for the cylinder).

It will also be obvious that the principal terms of the harmonics of the second order will be of the second degree in the radii.

The values of the coefficients can be calculated with ease, as with the cylinder, but we wait to see which of them are involved in the couple.

Writing the external field on the a, b shell in the form

$$\Omega_0 = Y_1 \cos pt + Z_1 \sin pt,$$

the couple will be (xii.)

$$\frac{1.3^2 p\sigma}{2.2\,\Delta_1} \iint dS . \left[Y_1 \frac{\partial Z_1}{\partial \phi} - Z_1 \frac{\partial Y_1}{\partial \phi} \right],$$

and the most important term omitted (that from the second harmonic) is of at least two degrees higher in powers of the radii.

Also,

$$Y_1 = {}^{\backprime}A \sin\theta \cos\phi + {}^{\backprime}B \sin\theta \sin\phi,$$
$$Z_1 = {}^{\backprime}D \sin\theta \cos\phi + {}^{\backprime}E \sin\theta \sin\phi,$$

so that the couple is

$$\frac{9p\sigma}{4\Delta_1} \iint dS\, [({}^{\backprime}A \cos\phi + {}^{\backprime}B \sin\phi)(-{}^{\backprime}D \sin\phi + {}^{\backprime}E \cos\phi)$$
$$- (-{}^{\backprime}A \sin\phi + {}^{\backprime}B \cos\phi)({}^{\backprime}D \cos\phi + {}^{\backprime}E \sin\phi)] \sin^2\theta,$$

or

$$\frac{9p\sigma}{4\Delta_1} \int_0^\pi \delta\theta . a^2 \sin^3\theta \int_0^{2\pi} \delta\phi \,\{({}^{\backprime}A{}^{\backprime}E - {}^{\backprime}B{}^{\backprime}D)(\sin^2\phi + \cos^2\phi)\},$$

or

$$\frac{9p\sigma}{4\Delta_1} . \tfrac{4}{3} a^2 . 2\pi ({}^{\backprime}A{}^{\backprime}E - {}^{\backprime}B{}^{\backprime}D) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(xiv.)}.$$

On reference to the values of ${}^{\backprime}A$, &c., it will be seen that to fourth powers of radii,

$${}^{\backprime}A{}^{\backprime}E - {}^{\backprime}B{}^{\backprime}D = -\frac{2Ia}{\rho^2} [cE' + 2bD'] \frac{aa'^2}{d^3},$$

also

$$D' = -\frac{12\pi p a'^2 \sigma' c}{\rho'^2 \Delta'_1} I + \text{terms three degrees higher,}$$

$$E' = -\frac{12\pi p a'^2 \sigma' b'}{\rho'^2 \Delta'_1} I \ldots$$

Hence the couple increasing ϕ is

$$\frac{6\pi p a^2 \sigma}{\Delta_1} \cdot \frac{21 a^2 a'^2}{\rho^2 d^3} \cdot \frac{12\pi p a'^2 \sigma'}{\rho'^2 \Delta'_1} I \, (cb' + 2bc)$$

or

$$\frac{144\pi^2 p^2 \sigma\sigma' a^4 a'^4 c}{\rho^2 \rho'^2 d^3 \Delta_1 \Delta'_1} (2b + b') I^2.$$

22. In obtaining the couple on the a', b' shell it will not do merely to interchange dashed and undashed letters, for the equations (xiii.) give when $r' = a'$ and $< d$,

$$r^{-1} = \frac{1}{d}\left[1 + \frac{a'}{d} \sin\theta' \cos\phi' \ldots\right],$$

so that

$$\frac{\sin\theta\cos\phi}{r^2} = \frac{1}{d^8}\left[1 + \frac{3a'}{d}\sin\theta'\cos\phi' \ldots\right]\left[-d + a'\sin\theta'\cos\phi'\right]$$

$$= -\frac{1}{d^2} - \frac{2a'}{d^3}\sin\theta'\cos\phi',$$

$$\frac{\sin\theta\sin\phi}{r^2} = \frac{a'}{d^3}\sin\theta'\sin\phi',$$

$$\frac{\cos\theta}{r^2} = \frac{a'}{d^3}\cos\theta'.$$

These equations may be obtained from (xiii.) by interchanging dashed and undashed letters, *leaving the sign of* d *unaltered*, with the exception of the term $-(1/d^2)$ in $(\sin\theta\cos\phi)/r^2$, and as this term does not appear afterwards (being constant over the sphere), the exception is negligible.

The subsequent work does not introduce d afresh, it only makes use of the formulæ we have obtained, and thus it will be seen that the final couple on the a', b' shell is got by the changing of dashed and undashed letters, leaving the sign of d unaltered; it is

$$\frac{144\pi^2 p^2 \sigma\sigma' a^4 a'^4 c}{\rho^2 \rho'^2 d^3 \Delta_1 \Delta'_1} (b + 2b') I^2.$$

If we write h for $\frac{1}{2}(b + b')$, the mean of distances of the centres from 0, the couples are

$$\frac{72\pi^2 p^2 \sigma\sigma' a^4 a'^4 c}{\rho^2 \rho'^2 d^3 \Delta_1 \Delta'_1} (6h \pm d) I^2.$$

23. Hence

(α.) If σ, σ' or c vanish, the couple vanishes, as might have been expected.

(β.) The signs of the couples fall into three cases :—

(i.) When $6h$ is positive and $> d$, both couples are positive, and tend to increase ϕ.

Fig. 36.

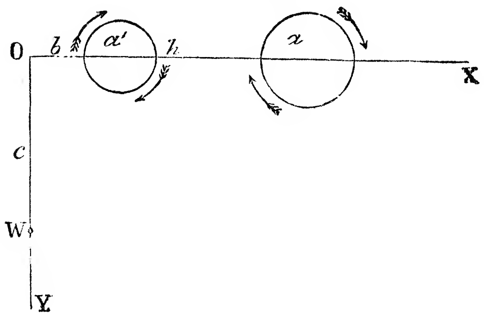
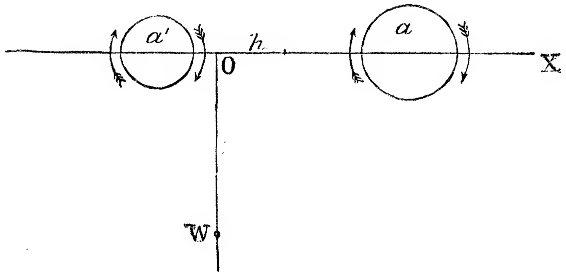
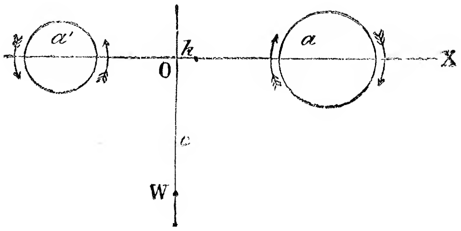


Fig. 37.



(ii.) When $6h$ is numerically less than d, the a, b couple is positive, and the a', b' negative.

Fig. 38.



(iii.) When $6h$ is negative and numerically $> d$, both couples are negative.

Fig. 39.

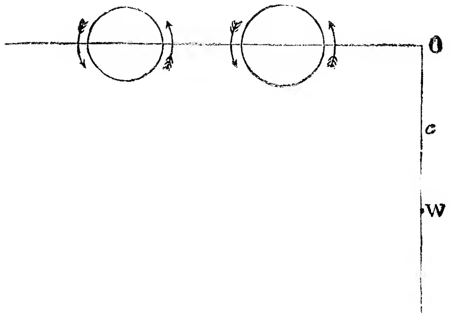
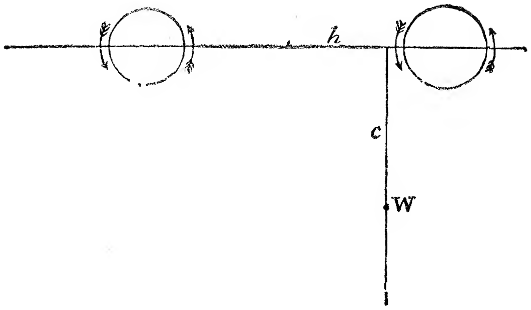
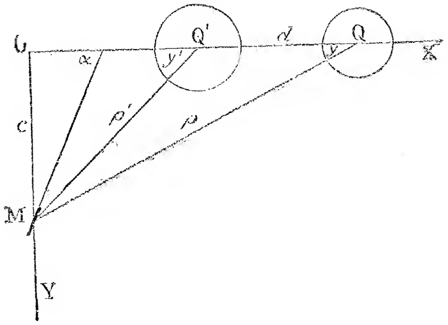


Fig. 40.



The Couples on two Spherical Shells, produced by a Filament of infinite length, perpendicular to the Line joining their Centres, and alternately Magnetised in a direction at right angles to its length.

Fig. 41.



24. Let the same axes be taken as before, and let the direction of magnetisation make α with the plane XOZ. The filament is equivalent to an electromagnet formed by currents

$$\frac{K}{\kappa}\cos pt,$$

parallel to OZ at $x = 0$, $y = c$, and

$$-\frac{K}{\kappa}\cos pt$$

at $x = -\kappa\sin\alpha$, $y = c - \kappa\cos\alpha$, in the limit when κ is small; the strength of magnetisation will then be $K\cos pt$.

As before, a coefficient of magnetic potential

$$If(b, b', c)$$

due to the current $I\cos pt$, will become

$$K\left[-\sin\alpha\left(\frac{\partial f}{\partial b} + \frac{\partial f}{\partial b'}\right) + \cos\alpha\frac{\partial f}{\partial c}\right]$$

due to the electromagnet, and if

$$\left.\begin{aligned}\rho\cos\gamma &= b\\ \rho\sin\gamma &= c\end{aligned}\right\}$$

then on reference to previous work it will be found that whereas we had for ${}^{\backprime}A$ due to $I\cos pt$,

$$= \frac{2Iac}{\rho^2} - \text{fourth powers},$$

now we shall have

$${}^{\backprime}A = -\frac{2K\alpha}{\rho^2}\left[\cos\alpha + 2\sin\gamma\sin\overline{\alpha-\gamma}\right] \quad . \quad . \quad .$$

So, too, we shall have instead of

$$ {}^{\backprime}\mathrm{B} = -\frac{2\mathrm{I}ab}{\rho^2} \ldots, $$

$$ {}^{\backprime}_{,}\mathrm{B} = -\frac{2\mathrm{K}a}{\rho^2}\left[-\sin\alpha + 2\cos\gamma\sin\overline{\alpha-\gamma}\right], $$

and

$$ {}^{\backprime}_{,}\mathrm{E} = \frac{aa'^2}{d^3}\,{}_{,}\mathrm{E}' $$

(denoting by suffix ($_{,}$) that we have an electromagnet, not a current).

Also

$$ \mathrm{E}' = -\frac{12\pi pa'^2\sigma'b'}{\Delta'_1\rho'^2}\,\mathrm{I}. $$

Therefore,

$$ {}_{,}\mathrm{E}' = -\frac{12\pi pa'^2\sigma'}{\Delta'_1\rho'^2}\,\mathrm{K}\left[-\sin\alpha + 2\cos\gamma'\sin\overline{\alpha-\gamma'}\right], $$

and similarly,

$$ {}^{\backprime}_{,}\mathrm{D} = -\frac{2aa'^2}{d^2}\,{}_{,}\mathrm{D}', $$

and

$$ {}_{,}\mathrm{D}' = -\frac{12p\pi a'^2\sigma'}{\Delta'_1\rho'^2}\,\mathrm{K}\left[\cos\alpha + 2\sin\gamma'\sin\overline{\alpha-\gamma'}\right]. $$

But the couple tending to increase ϕ on the a, b shell is, by (xiv.),

$$ \frac{6\pi pa^2\sigma}{\Delta_1}\left({}^{\backprime}_{,}\mathrm{A}\,{}^{\backprime}_{,}\mathrm{E} - {}^{\backprime}_{,}\mathrm{B}\,{}^{\backprime}_{,}\mathrm{D}\right), $$

or

$$ \frac{6\pi pa^2\sigma}{\Delta_1}\,\frac{aa'^2}{d^3}\left({}^{\backprime}_{,}\mathrm{A}\,{}_{,}\mathrm{E}' + 2\,{}^{\backprime}_{,}\mathrm{B}\,{}_{,}\mathrm{D}'\right), $$

or

$$ + \frac{6\pi pa^3a'^2}{\Delta_1 d^3}\cdot\frac{2\mathrm{K}a}{\rho^2}\cdot\frac{12p\pi a'^2\sigma'}{\Delta'_1\rho'^2}\,\mathrm{K}\cdot f(\alpha,\gamma,\gamma'), $$

where

$$ \begin{aligned} f(\alpha,\gamma,\gamma') &= \left[\cos\alpha + 2\sin\gamma\sin\overline{\alpha-\gamma}\right]\left[-\sin\alpha + 2\cos\gamma'\sin\overline{\alpha-\gamma'}\right] \\ &\quad + 2\left[-\sin\alpha + 2\cos\gamma\sin\overline{\alpha-\gamma}\right]\left[\cos\alpha + 2\sin\gamma'\sin\overline{\alpha-\gamma'}\right] \\ &= -3\sin\alpha\cos\alpha + \sin\overline{\alpha-\gamma}\left[4\cos\alpha\cos\gamma - 2\sin\alpha\sin\gamma\right] \\ &\quad + \sin\overline{\alpha-\gamma'}\left[2\cos\alpha\cos\gamma' - 4\sin\alpha\sin\gamma\right] \\ &\quad + 2\sin\overline{\alpha-\gamma}\sin\overline{\alpha-\gamma'}\left[2\sin\gamma\cos\gamma' + 4\cos\gamma\sin\gamma'\right] \\ &= -\tfrac{3}{2}\sin 2\alpha + \sin\overline{\alpha-\gamma}\left[\cos\overline{\alpha-\gamma} + 3\cos\overline{\alpha+\gamma}\right] \\ &\qquad\qquad + \sin\overline{\alpha-\gamma'}\left[3\cos\overline{\alpha+\gamma'} - \cos\overline{\alpha-\gamma'}\right] \\ &\quad + \left[\cos\overline{\gamma-\gamma'} - \cos 2\,\overline{\alpha-\gamma-\gamma'}\right]\left[3\sin\overline{\gamma+\gamma'} + \sin\overline{\gamma'-\gamma}\right]. \end{aligned} $$

On multiplying out and replacing products of sines and cosines by sines and cosines of added or subtracted angles most of the terms cancel, and we are left with

$$\tfrac{1}{2} \sin 2 (\gamma' - \gamma) + \tfrac{3}{2} \sin 2 (\alpha - \gamma - \gamma').$$

Thus the couple on the a, b sphere is

$$\mathrm{K}^2 \frac{72\pi^2 p^2 a^4 a'^4 \sigma\sigma'}{\rho^2 \rho'^2 \Delta_1 \Delta'_1 d^3} [\sin 2 \overline{\gamma' - \gamma} + 3 \sin 2 (\alpha - \gamma - \gamma')],$$

and on the a', b' sphere,

$$\mathrm{K}^2 \frac{72\pi^2 p^2 a^4 a'^4 \sigma\sigma'}{\rho^2 \rho'^2 \Delta_1 \Delta'_1 d^3} [- \sin 2 \overline{\gamma' - \gamma} + 3 \sin 2 (\alpha - \gamma' - \gamma)].$$

25. From this we see that

(α.) If the couples on the two shells be equal and opposite

$$(\alpha - \gamma - \gamma') = 0 \quad \text{or} \quad \pm \tfrac{1}{2} \pi,$$

i.e.,

$$\alpha = \gamma + \gamma' \quad \text{or} \quad \gamma + \gamma' \pm \tfrac{1}{2} \pi.$$

(β.) The couples will not vanish when $c = 0$ (and $\gamma = \gamma' = 0$), unless in addition $\alpha = 0$ or $\pm \frac{1}{2} \pi$ (in which case there is by symmetry obviously no couple).

(γ.) We may take as an example

$$\gamma = 30^\circ, \qquad \gamma' = 60^\circ,$$

and the couples will be

$$\frac{72\pi^2 p^2 a^4 a'^4 \sigma'\sigma}{\rho^2 \rho'^2 \Delta_1 \Delta'_1 d^3} [\pm \frac{\sqrt{3}}{2} - 3 \sin 2\alpha],$$

the upper sign referring to the a, b shell.

The bracket will be positive when $\alpha = - 45^\circ$ say.

The bracket will be positive and negative when $\alpha = 0$ or 90°.

Fig. 42.

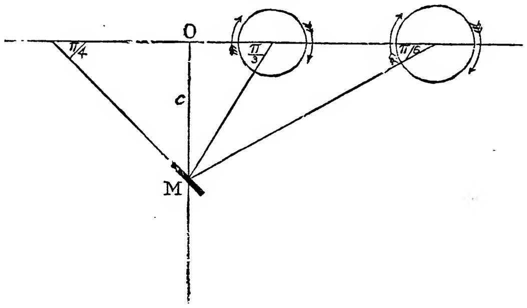
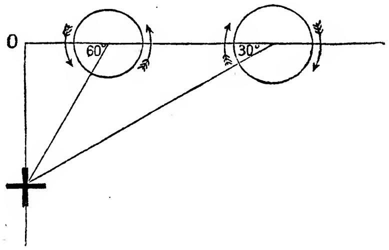
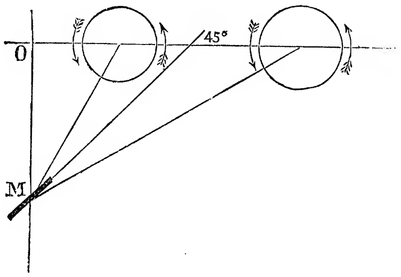


Fig. 43.



The bracket will be negative when $\alpha = + 45^\circ$.

Fig. 44.



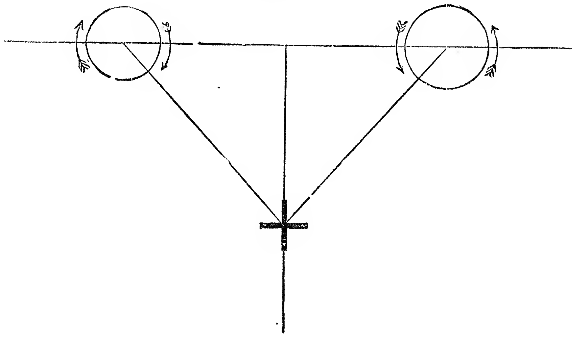
(δ.) In confirmation it may be noticed that if $\gamma = \pi - \gamma'$ (*i,e.*, when the system is symmetrical to OY), and $\alpha = 0^\circ$ or 90°, the couples are equal and opposite.

The couples are then

$$\mp \mathrm{K}^2 \frac{72\pi^2 p^2 a^4 a'^4 \sigma\sigma'}{\rho^2 \rho'^2 \Delta_1 \Delta'_1 d^3} \sin 2\gamma,$$

and are negative and positive on the a, b and a', b' shells respectively, for all possible values of γ, for which the a, b shell is to the right of a', b'.

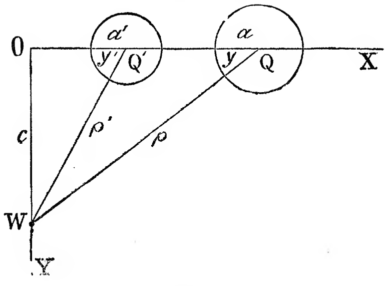
Fig. 45.



The Couples on Two Spherical Shells in the presence of a Magnetic Pole of strength H cos *pt.*

26. Take OX through the centres of the spheres and OY through the pole, whose distance from O is c.

Fig. 46.



Then, with the same notation as before, the magnetic potential due to the single pole will be, on the a, b shell,

$$\Omega_o = \frac{\mathrm{H}\cos pt}{[\rho^2 + a^2 + 2\rho a \sin\theta \cos\overline{\phi+\gamma}]^{\frac{1}{2}}}$$

$$= \frac{\mathrm{H}\cos pt}{\rho}\left[1 - \frac{a}{\rho}\sin\theta\cos\overline{\gamma+\phi} + \text{second harmonics}\right].$$

The equations (xiii.) now give

$$\left.\begin{aligned} {}^{\backprime}\mathrm{A} &= -\frac{\mathrm{H}a}{\rho^2}\cos\gamma + \text{fourth powers of } a \\ {}^{\backprime}\mathrm{B} &= \quad \frac{\mathrm{H}a}{\rho^2}\sin\gamma + \ldots \\ {}^{\backprime}\mathrm{D} &= -\frac{2aa'^2}{d^3}\mathrm{D}' \\ {}^{\backprime}\mathrm{E} &= +\frac{aa'^2}{d^3}\mathrm{E}' \end{aligned}\right\} \quad \ldots\ldots \quad \text{(xv.)},$$

and since in Ω_0 there was no term $\cos\theta$, the coefficients ${}^{\backprime}\mathrm{C}$, ${}^{\backprime}\mathrm{F}$ will be zero.

Also

$$\mathrm{D} = \frac{2\pi pa}{\Delta_1}[3\sigma\,.\,{}^{\backprime}\mathrm{A}] + \text{fourth powers},$$

$$\mathrm{E} = \frac{2\pi pa}{\Delta_1}[3\sigma\,.\,{}^{\backprime}\mathrm{B}]\ldots$$

Therefore

$$\mathrm{D}' = \frac{2\pi pa'}{\Delta'_1}3\sigma'\,.\left[-\frac{\mathrm{H}a'}{\rho'}\cos\gamma'\right]\ldots,$$

$$\mathrm{E}' = \frac{2\pi pa'}{\Delta'_1}3\sigma'\,.\left[+\frac{\mathrm{H}a'}{\rho'^2}\sin\gamma'\right]\ldots$$

The couple on the a, b shell has been proved to be (xiv.)

$$\frac{6\pi pa^2\sigma}{\Delta_1}({}^{\backprime}\mathrm{A}{}^{\backprime}\mathrm{E} - {}^{\backprime}\mathrm{B}{}^{\backprime}\mathrm{D}).$$

This is equal to

$$\frac{6\pi pa^2\sigma}{\Delta_1}\frac{aa'^2}{d^3}[{}^{\backprime}\mathrm{A}\,.\,\mathrm{E}' + 2{}^{\backprime}\mathrm{B}\,.\,\mathrm{D}'],$$

or

$$\frac{6\pi pa^2\sigma}{\Delta_1}\frac{aa'^2}{d^3}\frac{\mathrm{H}a}{\rho^2}\frac{6\pi pa'\sigma'}{\Delta'_1}\frac{\mathrm{H}a'}{\rho'^2}[-\cos\gamma\sin\gamma' - 2\sin\gamma\cos\gamma'],$$

or

$$-\mathrm{H}^2\frac{36\pi^2p^2a^4a'^4c\sigma\sigma'}{\rho^3\rho'^3\Delta_1\Delta'_1 d^3}(b + 2b').$$

The couple on the a', b' shell is

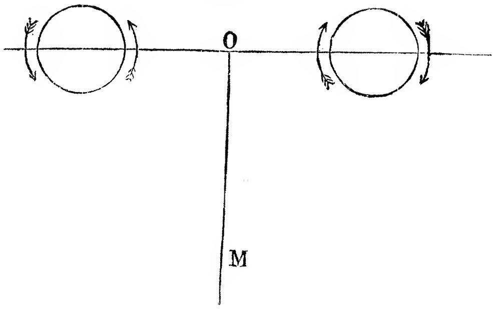
$$- \mathrm{H}^2 \frac{36\pi^2 p^2 a^4 a'^4 c\sigma\sigma'}{\rho^3 \rho'^3 \Delta_1 \Delta'_1 d^3} (2b + b').$$

27. Thus we see that

(α.) The couples vanish (as they should) when $c = 0$.

(β.) When $b = - b'$ and $a = a'$, the figure is symmetrical to plane YOZ, and the couples will be equal and opposite (as they should): that on the a, b shell being positive if d be positive.

Fig. 47.



(γ.) For other cases the discussion of sign is similar to that for the current I cos *pt*, and there are three cases :—

I. When b and b' are positive, the couples are both negative, as also if b' be negative and $\frac{1}{2}(b + b') > d/6$.

Fig. 48.

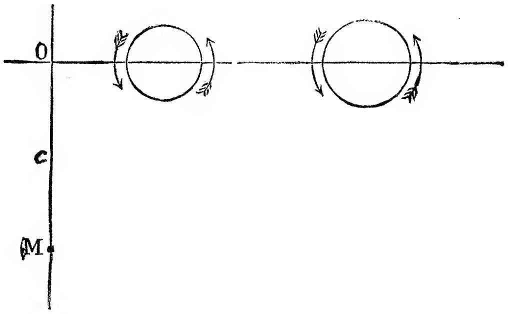
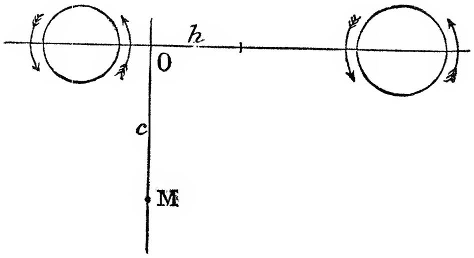
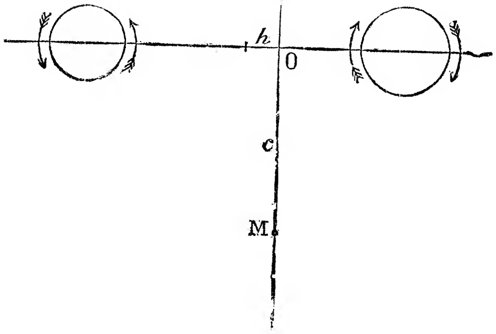


Fig. 49.



II. When $\frac{1}{2}(b + b') < d/6$ numerically, the signs are +, —.

Fig. 50.



III. When $\frac{1}{2}(b + b')$ is negative, and is numerically greater than $d/6$, the couples are both positive.

Fig. 51.

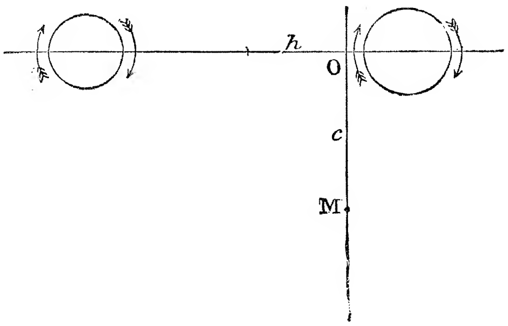
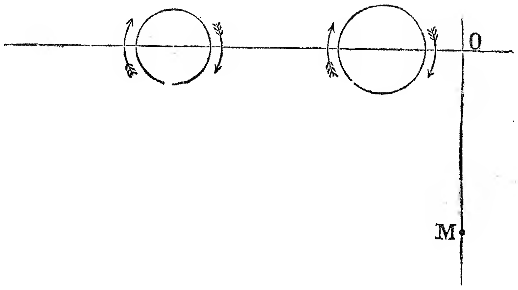


Fig. 52.

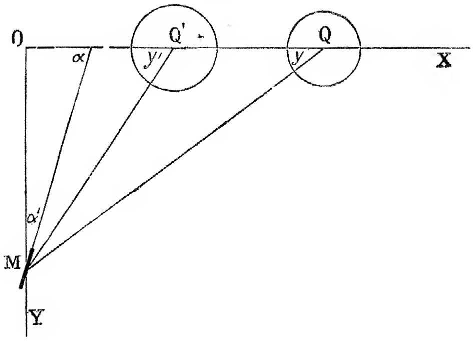


The Couples on two Spherical Shells in the presence of a small Magnet of Moment K cos *pt, whose Axis cuts the Line joining the Centres of the Spheres at an angle* α.

28. We regard the magnetic particle as having a pole of strength $(\mathrm{K}/\kappa)\cos pt$ at $x = 0$, $y = c$, and a pole of strength $-(\mathrm{K}/\kappa)\cos pt$ at

$$x = -\kappa \sin \alpha', \quad y = c + \kappa \cos \alpha', \quad \text{where} \quad \alpha' = (\pi/2) - \alpha.$$

Fig. 53.



If a coefficient of magnetic potential due to a single pole at $(0, c)$ of strength $\mathrm{H}\cos pt$, be

$$\mathrm{H}f(b, b', c)$$

that due to the magnet will be the limit when κ is small of

$$(\mathrm{K}/\kappa)f(b, b', c) - (\mathrm{K}/\kappa)f(b + \kappa \sin \alpha', b' + \kappa \sin \alpha', c + \kappa \cos \alpha'),$$

i.e., will be

$$-\mathrm{K}\left[\left(\frac{\partial f}{\partial b} + \frac{\partial f}{\partial b'}\right)\sin \alpha' + \frac{\partial f}{\partial c}\cos \alpha'\right].$$

Now we had (xv.)

$$ {}^{\backprime}\mathrm{A} = -\frac{\mathrm{H}a}{\rho^2}\cos\gamma = -\frac{\mathrm{H}ab}{\rho^3}. $$

Thus, distinguishing coefficients arising from the magnet by a suffix ($_{,}$) as distinct from the magnetic pole, we have

$$ \begin{aligned} {}^{\backprime}_{,}\mathrm{A} &= -\mathrm{K}a\left[\left(-\frac{1}{\rho^3}+\frac{3b^2}{\rho^5}\right)\sin\alpha' + \frac{3bc}{\rho^5}\cos\alpha'\right] \\ &= -\frac{\mathrm{K}a}{\rho^3}\,[\sin\alpha' + 3\cos\gamma\sin\overline{\alpha'+\gamma}] \end{aligned} $$

So too from ${}^{\backprime}\mathrm{B} = \mathrm{H}ac/\rho^3$ (xv.)

$$ {}^{\backprime}_{,}\mathrm{B} = \frac{\mathrm{K}a}{\rho^3}\,[-\cos\alpha' + 3\sin\gamma\sin\overline{\alpha'+\gamma}]. $$

We had also

$$ \begin{aligned} {}^{\backprime}\mathrm{D} &= \frac{2aa'^2}{d^3}\cdot\frac{6\pi pa'\sigma'}{\Delta'_1}\,\frac{\mathrm{H}a'b'}{\rho'^3} + \text{higher powers}, \\ {}^{\backprime}\mathrm{E} &= \frac{aa'^2}{d^3}\cdot\frac{6\pi pa'\sigma'}{\Delta_1'}\,\frac{\mathrm{H}a'c}{\rho'^3}. \end{aligned} $$

Hence

$$ \begin{aligned} {}^{\backprime}_{,}\mathrm{D} &= \mathrm{K}\,\frac{12\pi paa'^4\sigma'}{d^3\Delta'_1\rho'^3}\,[-\sin\alpha' + 3\cos\gamma'\sin\overline{\alpha'+\gamma'}]\ldots \\ {}^{\backprime}_{,}\mathrm{E} &= \mathrm{K}\,\frac{6\pi paa'^4\sigma'}{d^3\Delta'_1\rho'^3}\,[-\cos\alpha' + 3\sin\gamma'\sin\overline{\alpha'+\gamma'}]\ldots \end{aligned} $$

Thus the couple

$$ \begin{aligned} &= \frac{6\pi pa^2\sigma}{\Delta_1}\,({}^{\backprime}_{,}\mathrm{A}\,{}^{\backprime}_{,}\mathrm{E} - {}^{\backprime}_{,}\mathrm{B}\,{}^{\backprime}_{,}\mathrm{D}), \\ &= \frac{6\pi pa^2\sigma}{\Delta_1}\cdot\frac{\mathrm{K}^2a}{\rho^3}\cdot\frac{6\pi paa'^4\sigma'}{d^3\Delta'_1\rho'^3}\,f(\alpha'\,\gamma,\gamma'), \end{aligned} $$

where

$$ \begin{aligned} f(\alpha',\gamma,\gamma') \equiv &-[-\sin\alpha' + 3\cos\gamma\sin\overline{\alpha'+\gamma}]\,[-\cos\alpha' + 3\sin\gamma'\sin\overline{\alpha'+\gamma'}] \\ &-[-\cos\alpha' + 3\sin\gamma\sin\overline{\alpha'+\gamma}]\,[-2\sin\alpha' + 6\cos\gamma'\sin\overline{\alpha'+\gamma'}] \\ = &-\tfrac{1}{4}\{(\sin\alpha' + 3\sin\overline{\alpha'+2\gamma})\,(\cos\alpha' - 3\cos\overline{\alpha'+2\gamma'}) \\ &+ (\cos\alpha' - 3\cos\overline{\alpha'+2\gamma})\,(\sin\alpha' + 3\sin\overline{\alpha'+2\gamma'})\}. \end{aligned} $$

On multiplying and continuing the practice of replacing products by sines and cosines of sums or differences, we get

$$ \begin{aligned} = -\tfrac{3}{8}\,[&\sin 2\alpha' + 3\,(\sin 2\gamma + \sin 2\gamma') + (\sin 2\,\overline{\alpha'+\gamma'} - \sin 2\overline{\alpha'+\gamma}) \\ &- 3\sin 2\,(\gamma'-\gamma) - 9\sin\overline{2\alpha'+2\gamma+2\gamma'}]. \end{aligned} $$

Expressed in terms of α, the couple on the two shells will be

$$
\begin{aligned}
&- \mathrm{K}^2 \frac{27\pi^2 p^2 a^4 a'^4 \sigma\sigma'}{2\Delta_1 \Delta'_1\, d^3 \rho^3 \rho'^3}\, \phi(\alpha, \gamma, \gamma') \\
&- \mathrm{K}^2 \frac{27\pi^2 p^2 a^4 a'^4 \sigma\sigma'}{2\Delta_1 \Delta'_1\, d^3 \rho^3 \rho'^3}\, \phi(\alpha, \gamma', \gamma),
\end{aligned}
$$

where

$$
\begin{aligned}
\phi(\alpha, \gamma, \gamma') \equiv \sin 2\alpha &+ 3(\sin 2\gamma + \sin 2\gamma') + \sin \overline{2\alpha - 2\gamma'} \\
&- \sin \overline{2\alpha - 2\gamma} - 3 \sin \overline{2\gamma' - 2\gamma} - 9 \sin \overline{2\alpha - 2\gamma - 2\gamma'}.
\end{aligned}
$$

29. Hence we have

(α.) If $\gamma' = \pi - \gamma$,

$$\phi(\alpha, \gamma, \gamma') = 3 \sin 4\gamma + 2 \sin 2\gamma \cos 2\alpha - 8 \sin 2\alpha.$$

For the couples to be equal and opposite we must have $\alpha = 0$ or $\frac{1}{2}\pi$, and then

$$\phi(\alpha, \gamma, \gamma') = 3 \sin 4\gamma \pm 2 \sin 2\gamma.$$

If $\gamma = \frac{1}{3}\pi$ (fig. 54), the values of $\phi(\alpha, \gamma, \gamma')$ are negative, both for $\alpha = 0$ and $\alpha = \frac{1}{2}\pi$; if $\gamma = \frac{1}{6}\pi$ (fig. 55), both values of $\phi(\alpha, \gamma, \gamma')$ are positive.

Fig. 54.

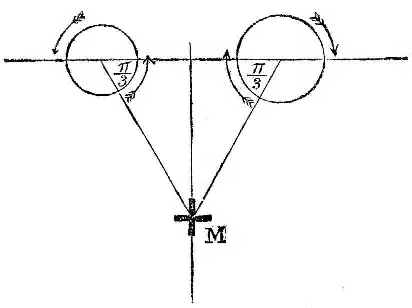
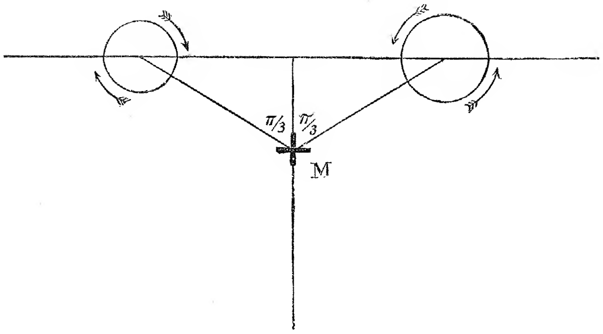


Fig. 55.



(β.) If we take $\gamma = 30°$, $\gamma' = 60°$, we find that

$$
\begin{aligned}
f(\alpha, \gamma, \gamma') &= -\tfrac{3}{8}\left[\frac{3\sqrt{3}}{2} + 9 \sin 2\alpha\right] \\
f(\alpha, \gamma', \gamma) &= -\tfrac{3}{8}\left[\frac{9\sqrt{3}}{2} + 11 \sin 2\alpha\right].
\end{aligned}
$$

The former is negative when α increases from 0° to about 98° 23′, and positive thence to 171° 37′, being negative to 180°. The latter is negative from 0° to 112° 34′, positive thence to 157° 26′, and afterwards negative to 180°.

In fig. 56, $\alpha = 45°$; in fig. 57, $\alpha = 105°$; in fig. 58, $\alpha = 135°$; and in fig. 59, $\alpha = 165°$.

Fig. 56.

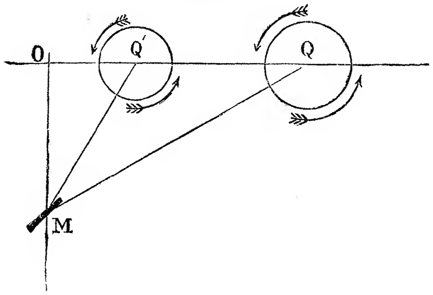


Fig. 57.

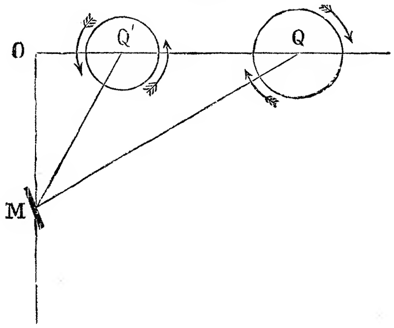


Fig. 58.

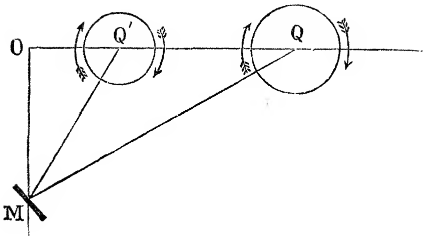


Fig. 59.